

A REFLEXIVE HI SPACE WITH THE HEREDITARY INVARIANT SUBSPACE PROPERTY

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ABSTRACT. A reflexive hereditarily indecomposable Banach space $\mathfrak{X}_{\text{ISP}}$ is presented, such that for every Y infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and every bounded linear operator $T : Y \rightarrow Y$, the operator T admits a non-trivial closed invariant subspace.

INTRODUCTION

The invariant subspace problem asks whether every bounded linear operator on an infinite dimensional separable Banach space admits a non-trivial closed invariant subspace. A classical result of M. Aronszajn and K.T. Smith [9] asserts that the problem has a positive answer for compact operators. This result was extended by V. Lomonosov [18] for operators on complex Banach spaces that commute with a non-trivial compact operator. Recently G. Sirotkin [27] has presented a version of Lomonosov's theorem for real spaces. It is also known that the problem, in its full generality, has a negative answer. Indeed P. Enflo [13] and subsequently C. J. Read [23],[24] have provided several examples of operators on non-reflexive Banach spaces that do not admit a non-trivial invariant subspace. Also recently a non-reflexive hereditarily indecomposable (HI) Banach space \mathfrak{X}_K with the “scalar plus compact” property has been constructed [7]. This is a \mathcal{L}_∞ space with separable dual, resulting from a combination of HI techniques with the fundamental J. Bourgain and F. Delbaen construction [10]. As consequence, the space \mathfrak{X}_K satisfies the Invariant Subspace Property (ISP). All the above results provide no information in either direction within the class of reflexive Banach spaces.

The aim of the present work is to construct a reflexive Banach space $\mathfrak{X}_{\text{ISP}}$ with the hereditary ISP. Namely, every infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ satisfies ISP, a property which is unknown for the aforementioned

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space \mathfrak{X}_K . It is notable that no subspace of $\mathfrak{X}_{\text{ISP}}$ has the “scalar plus compact” property. More precisely, the strictly singular operators¹ on every subspace Y of $\mathfrak{X}_{\text{ISP}}$ form a non separable ideal (in particular, the strictly singular non-compact are non-separable).

The space $\mathfrak{X}_{\text{ISP}}$ is a hereditarily indecomposable space and every operator $T \in \mathcal{L}(\mathfrak{X}_{\text{ISP}})$ is of the form $T = \lambda I + S$ with S strictly singular. We recall that there are strictly singular operators in Banach spaces without non-trivial invariant subspaces [25]. On the other hand, there are spaces where the ideal of strictly singular operators does not coincide with the corresponding one of compact operators and every strictly singular operator admits a non-trivial invariant subspace. The most classical result in this direction, due to V. Milman [19], concerns the strictly singular operators in $L^p[0, 1]$, $1 \leq p < \infty$, $C[0, 1]$. This is a consequence of Lomonosov-Sirotkin theorem and the fact that the composition TS is a compact operator, for any T, S strictly singular operators, on any of the above spaces. In [2], Tsirelson like spaces satisfying similar properties are presented. The possibility of constructing a reflexive space with ISP without the “scalar plus compact” property emerged from an earlier version of [2].

The following describes the main properties of the space $\mathfrak{X}_{\text{ISP}}$.

Theorem. There exists a reflexive space $\mathfrak{X}_{\text{ISP}}$ with a Schauder basis $\{e_n\}_{n \in \mathbb{N}}$ satisfying the following properties.

- (i) The space $\mathfrak{X}_{\text{ISP}}$ is hereditarily indecomposable.
- (ii) Every seminormalized weakly null sequence $\{x_n\}_{n \in \mathbb{N}}$ has a subsequence generating either ℓ_1 or c_0 as a spreading model. Moreover every infinite dimensional subspace Y of $\mathfrak{X}_{\text{ISP}}$ admits both ℓ_1 and c_0 as spreading models.
- (iii) For every Y infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and every $T \in \mathcal{L}(Y, \mathfrak{X}_{\text{ISP}})$, $T = \lambda I_{Y, \mathfrak{X}_{\text{ISP}}} + S$ with S strictly singular.
- (iv) For every Y infinite dimensional subspace of $\mathfrak{X}_{\text{ISP}}$ the ideal $\mathcal{S}(Y)$ of the strictly singular operators is non separable.
- (v) For every Y subspace of $\mathfrak{X}_{\text{ISP}}$ and every Q, S, T in $\mathcal{S}(Y)$ the operator QST is compact. Hence for every $T \in \mathcal{S}(Y)$ either $T^3 = 0$ or T commutes with a non zero compact operator.
- (vi) For every Y infinite dimensional closed subspace of X and every $T \in \mathcal{L}(Y)$, T admits a non-trivial closed invariant subspace. In particular every $T \neq \lambda I_Y$, for $\lambda \in \mathbb{R}$ admits a non-trivial hyperinvariant subspace.

It is not clear to us if the number of operators in property (v) can be reduced. For defining the space $\mathfrak{X}_{\text{ISP}}$ we use classical ingredients like the

¹A bounded linear operator is called strictly singular, if its restriction on any infinite dimensional subspace is not an isomorphism.

coding function σ , the interaction between conditional and unconditional structure, but also some new ones which we are about to describe.

In all previous HI constructions, one had to use a mixed Tsirelson space as the unconditional frame on which the HI norm is built. Mixed Tsirelson spaces appeared with Th. Schlumprecht space [26], twenty years after Tsirelson construction [28]. They became an inevitable ingredient for any HI construction, starting with the W.T. Gowers and B. Maurey celebrated example [16], and followed by myriads of others [4],[8] etc. The most significant difference in the construction of $\mathfrak{X}_{\text{ISP}}$ from the classical ones, is that it uses as an unconditional frame the Tsirelson space itself.

As it is clear to the experts, HI constructions based on Tsirelson space, are not possible if we deal with a complete saturation of the norm. Thus the second ingredient involves saturation under constraints. This method was introduced by E. Odell and Th. Schlumprecht [20],[21] for defining heterogeneous local structure in HI spaces, a method also used in [2]. By saturation under constraints we mean that the operations $(\frac{1}{2^n}, \mathcal{S}_n)$ (see Remark 1.5) are applied on very fast growing families of averages, which are either α -averages or β -averages. The α -averages have been also used in [20],[21], while β -averages are introduced to control the behaviour of special functionals. It is notable that although the α, β -averages do not contribute to the norm of the vectors in $\mathfrak{X}_{\text{ISP}}$, they are able to neutralize the action of the operations $(\frac{1}{2^n}, \mathcal{S}_n)$ on certain sequences and thus c_0 spreading models become abundant. This significant property yields the structure of $\mathfrak{X}_{\text{ISP}}$ described in the above theorem.

Let us briefly describe some further structural properties of the space $\mathfrak{X}_{\text{ISP}}$.

The first and most crucial one is that for a (n, ε) special convex combination (see Definition 1.9) $\sum_{i \in F} c_i x_i$, with $\{x_i\}_{i \in F}$ a finite normalized block sequence, we have that

$$\left\| \sum_{i \in F} c_i x_i \right\| \leq \frac{6}{2^n} + 12\varepsilon$$

This evaluation is due to the fact that the space is built on Tsirelson space and differs from the classical asymptotic ℓ_1 HI spaces (i.e. [4],[8]) where seminormalized (n, ε) s.c.c. appear everywhere. A consequence of the above, is that the frequency of the appearance of RIS sequences is significantly increased, which yields the following. Every strictly singular operator maps sequences generating c_0 spreading models to norm null ones. Furthermore for every two strictly singular operators $T, S : \mathfrak{X}_{\text{ISP}} \rightarrow \mathfrak{X}_{\text{ISP}}$ such that TS is non-compact and every weakly null sequence $\{x_n\}_n$ such that $\{TSx_n\}_n$ is not norm convergent, the following holds. Every spreading model generated by a subsequence of $\{TSx_n\}_n$ is c_0 . Combining the above properties we conclude property (v) of the above theorem.

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1. THE NORMING SET OF THE SPACE $\mathfrak{X}_{\text{ISP}}$

In this section we define the norming set W of the space $\mathfrak{X}_{\text{ISP}}$. This set is defined with the use of the sequence $\{\mathcal{S}_n\}_n$ which we remind below and also families of \mathcal{S}_n -admissible functionals.

As we have mentioned in the introduction, the set W will be a subset of the norming set W_T of the Tsirelson space.

The Schreier families. The Schreier families is an increasing sequence of families of finite subsets of the naturals, first appeared in [1], inductively defined in the following manner.

Set $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\}$ and $\mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leq \min F\}$.

Suppose that \mathcal{S}_n has been defined and set $\mathcal{S}_{n+1} = \{F \subset \mathbb{N} : F = \cup_{j=1}^k F_j, \text{ where } F_1 < \dots < F_k \in \mathcal{S}_n \text{ and } k \leq \min F_1\}$

If for $n, m \in \mathbb{N}$ we set $\mathcal{S}_n * \mathcal{S}_m = \{F \subset \mathbb{N} : F = \cup_{j=1}^k F_j, \text{ where } F_1 < \dots < F_k \in \mathcal{S}_m \text{ and } \{\min F_j : j = 1, \dots, k\} \in \mathcal{S}_n\}$, then it is well known that $\mathcal{S}_n * \mathcal{S}_m = \mathcal{S}_{n+m}$.

Notation. A sequence of vectors $x_1 < \dots < x_k$ in c_{00} is said to be \mathcal{S}_n -admissible if $\{\min \text{supp } x_i : i = 1, \dots, k\} \in \mathcal{S}_n$.

Let $G \subset c_{00}$. A vector $f \in G$ is said to be an average of size $s(f) = n$, if there exist $f_1, \dots, f_d \in G, d \leq n$, such that $f = \frac{1}{n}(f_1 + \dots + f_d)$.

A sequence $\{f_j\}_j$ of averages in G is said to be very fast growing, if $f_1 < f_2 < \dots$, $s(f_j) > (\max \text{supp } f_{j-1})^2$ and $s(f_j) > s(f_{j-1})$ for $j > 1$.

The coding function. Choose $L = \{\ell_k : k \in \mathbb{N}\}, \ell_1 > 2$ an infinite subset of the naturals such that:

- (i) For any $k \in \mathbb{N}$ we have that $\ell_{k+1} > 2^{2^{\ell_k}}$ and
- (ii) $\sum_{k=1}^{\infty} \frac{1}{2^{\ell_k}} < \frac{1}{1000}$.

Decompose L into further infinite subsets L_1, L_2 . Set

$$\begin{aligned} \mathcal{Q} = & \left\{ ((f_1, n_1), \dots, (f_m, n_m)) : m \in \mathbb{N}, \{n_k\}_{k=1}^m \subset \mathbb{N}, f_1 < \dots < f_m \in c_{00} \right. \\ & \left. \text{with } f_k(i) \in \mathbb{Q}, \text{ for } i \in \mathbb{N}, k = 1, \dots, m \right\} \end{aligned}$$

Choose a one to one function $\sigma : \mathcal{Q} \rightarrow L_2$, called the coding function, such that for any $((f_1, n_1), \dots, (f_m, n_m)) \in \mathcal{Q}$, we have that

$$\sigma((f_1, n_1), \dots, (f_m, n_m)) > 2^{n_m} \cdot \max \text{supp } f_m$$

Remark 1.1. For any $n \in \mathbb{N}$ we have that $\#L \cap \{n, \dots, 2^{2^n}\} \leq 1$.

The norming set. The norming set W is defined to be the smallest subset of c_{00} satisfying the following properties:

1. The set $\{\pm e_n\}_{n \in \mathbb{N}}$ is a subset of W , for any $f \in W$ we have that $-f \in W$, for any $f \in W$ and any I interval of the naturals we have that $If \in W$ and

W is closed under rational convex combinations. Any $f = \pm e_n$ will be called a functional of type 0.

2. The set W contains any functional f which is of the form $f = \frac{1}{2^n} \sum_{j=1}^d \alpha_j$, where $\{\alpha_j\}_{j=1}^d$ is an \mathcal{S}_n -admissible and very fast growing sequence of α -averages in W . If I is an interval of the naturals, then $g = \pm If$ is called a functional of type I_α , of weight $w(g) = n$.

3. The set W contains any functional f which is of the form $f = \frac{1}{2^n} \sum_{j=1}^d \beta_j$, where $\{\beta_j\}_{j=1}^d$ is an \mathcal{S}_n -admissible and very fast growing sequence of β -averages in W . If I is an interval of the naturals, then $g = \pm If$ is called a functional of type I_β , of weight $w(g) = n$.

4. The set W contains any functional f which is of the form $f = \frac{1}{2} \sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is an \mathcal{S}_1 -admissible special sequence of type I_α functionals. This means that $w(f_1) \in L_1$ and $w(f_j) = \sigma((f_1, w(f_1)), \dots, (f_{j-1}, w(f_{j-1})))$, for $j > 1$. If I is an interval of the naturals, then $g = \pm If$ is called a functional of type II with weights $\widehat{w}(g) = \{w(f_j) : \text{ran } f_j \cap I \neq \emptyset\}$.

We call an α -average any average $\alpha \in W$ of the form $\alpha = \frac{1}{n} \sum_{j=1}^d f_j$, $d \leq n$, where $f_1 < \dots < f_d \in W$.

We call a β -average any average $\beta \in W$ of the form $\beta = \frac{1}{n} \sum_{j=1}^d f_j$, $d \leq n$, where $f_1, \dots, f_d \in W$ are functionals of type II, with disjoint weights $\widehat{w}(f_j)$.

In general, we call a convex combination any $f \in W$ that is not of type 0, I_α , I_β or II.

For $x \in c_{00}$ define $\|x\| = \sup\{f(x) : f \in W\}$ and $\mathfrak{X}_{\text{ISP}} = \overline{(c_{00}(\mathbb{N}), \|\cdot\|)}$. Evidently $\mathfrak{X}_{\text{ISP}}$ has a bimonotone basis.

One may also describe the norm on $\mathfrak{X}_{\text{ISP}}$ with an implicit formula. Indeed, for some $x \in \mathfrak{X}_{\text{ISP}}$, we have that

$$\|x\| = \max \left\{ \|x\|_0, \|x\|_{II}, \sup \left\{ \frac{1}{2^n} \sum_{j=1}^d \|E_j x\|_{k_j}^\alpha \right\}, \sup \left\{ \frac{1}{2^n} \sum_{j=1}^d \|E_j x\|_{k_j}^\beta \right\} \right\}$$

where the inner suprema are taken over all $n \in \mathbb{N}$, all \mathcal{S}_n -admissible intervals $\{E_j\}_{j=1}^d$ of the naturals and $k_1 < \dots < k_d$ such that $k_j > (\max E_{j-1})^2$ for $j > 1$.

By $\|x\|_{II}$ we denote

$$\|x\|_{II} = \sup\{f(x) : f \in W \text{ is a functional of type II}\}$$

whereas for $j \in \mathbb{N}$, by $\|x\|_j^\alpha$ we denote

$$\|x\|_j^\alpha = \sup\{\alpha(x) : \alpha \in W \text{ is an } \alpha\text{-average of size } s(\alpha) = j\}$$

Similarly, by $\|x\|_j^\beta$ we denote

$$\|x\|_j^\beta = \sup\{\beta(x) : \beta \in W \text{ is a } \beta\text{-average of size } s(\beta) = j\}.$$

Remark 1.2. Very fast growing sequences of α -averages have been considered by E. Odell and Th. Schlumprecht in [20], [21] and were also used in

[2]. However, β -averages are a new ingredient, introduced to control the behaviour of type II functionals on block sequences. The β -averages can also be used to provide an alternative and simpler approach of the main result in [21].

As we have mentioned in the introduction, the $\|z\|_j^\alpha, \|z\|_j^\beta$, which are averages, do not contribute to the norm of the vector z . On the other hand, the $\{\|\cdot\|_j^\alpha\}_j, \{\|\cdot\|_j^\beta\}_j$ have a significant role for the structure of the space $\mathfrak{X}_{\text{ISP}}$.

Remark 1.3. The norming set W can be inductively constructed to be the union of an increasing sequence of subsets $\{W_m\}_{m=0}^\infty$ of c_{00} , where $W_0 = \{\pm e_n\}_{n \in \mathbb{N}}$ and if W_m has been constructed, then set W_{m+1}^α to be the closure of W_m under taking α -averages, $W_{m+1}^{I_\alpha}$ to be the closure of W_{m+1}^α under taking type I_α functionals, $W_{m+1}^{I_\beta}$ to be the closure of $W_{m+1}^{I_\alpha}$ under taking type I_β functionals, W_{m+1}^Π to be the closure of $W_{m+1}^{I_\beta}$ under taking type II functionals, W_{m+1}^β to be the closure of W_{m+1}^Π under taking β -averages and finally W_{m+1} to be the closure of W_{m+1}^β under taking rational convex combinations.

Tsirelson space. Tsirelson's initial definition [28] of the first Banach space not containing any $\ell_p, 1 \leq p < \infty$ or c_0 , concerned the dual of the so called Tsirelson norm which was introduced by T. Figiel and W. B. Johnson [15] and satisfies the following implicit formula.

$$\|x\|_T = \max \left\{ \|x\|_0, \sup \left\{ \frac{1}{2} \sum_{j=1}^d \|E_j x\|_T \right\} \right\}$$

where $x \in c_{00}$ and the inner supremum is taken over all successive subsets of the naturals $d \leq E_1 < \dots < E_d$. Tsirelson space T is defined to be the completion of $(c_{00}, \|\cdot\|_T)$. In the sequel by Tsirelson norm and Tsirelson space we will mean the norm and the corresponding space from [15].

As is well known, a norming set W_T of Tsirelson space is the smallest subset of c_{00} satisfying the following properties.

1. The set $\{\pm e_n\}_{n \in \mathbb{N}}$ is a subset of W_T , for any $f \in W_T$ we have that $-f \in W_T$, for any $f \in W_T$ and any E subset of the naturals we have that $Ef \in W_T$ and W_T is closed under rational convex combinations.
2. The set W_T contains any functional f which is of the form $f = \frac{1}{2} \sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is a \mathcal{S}_1 admissible sequence in W_T .

Remark 1.4. The following are well known facts about Tsirelson space.

- (i) The norming set W_T can be inductively constructed to be the union of an increasing sequence of subsets $\{W_T^m\}_{m=0}^\infty$ of c_{00} , in a similar manner as above.
- (ii) The set W'_T , which is the smallest subset of c_{00} satisfying the following properties, also is a norming set for Tsirelson space.

1. The set $\{\pm e_n\}_{n \in \mathbb{N}}$ is a subset of W'_T , for any $f \in W'_T$ we have that $-f \in W'_T$ and for any $f \in W'_T$ and any E subset of the naturals we have that $Ef \in W'_T$.
2. The set W'_T contains any functional f which is of the form $f = \frac{1}{2} \sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is a \mathcal{S}_1 admissible sequence in W'_T .

Remark 1.5. It is easy to check that the norming set W_T of Tsirelson space is closed under $(\frac{1}{2^n}, \mathcal{S}_n)$ operations, namely for any $f_1 < \dots < f_d$ in W_T \mathcal{S}_n -admissible, the functional $\frac{1}{2^n} \sum_{j=1}^d f_j \in W_T$. This explains that the norming set W of the space $\mathfrak{X}_{\text{ISP}}$ is a subset of W_T . Therefore Tsirelson space is the unconditional frame on which the norm of $\mathfrak{X}_{\text{ISP}}$ is built. As we mentioned in the introduction, $\mathfrak{X}_{\text{ISP}}$ is the first HI construction which uses Tsirelson space instead of a mixed Tsirelson one.

As it is shown in [11] (see also [12]), an equivalent norm on Tsirelson space is described by the following implicit formula. For $x \in c_{00}$ set

$$\|x\| = \max \left\{ \|x\|_0, \sup \left\{ \frac{1}{2} \sum_{j=1}^{2d} \|E_j x\| \right\} \right\}$$

where the inner supremum is taken over all successive subsets of the naturals $d \leq E_1 < \dots < E_{2d}$. Then, for any $\{c_k\}_{k=1}^n \subset \mathbb{R}$, the following holds.

$$(1) \quad \left\| \sum_{k=1}^n c_k e_k \right\|_T \leq \left\| \sum_{k=1}^n c_k e_k \right\| \leq 3 \left\| \sum_{k=1}^n c_k e_k \right\|_T$$

Remark 1.6. A norming set $W_{(T, \|\cdot\|)}$ for $(T, \|\cdot\|)$ is also defined in a similar manner as W_T .

Special convex combinations. Next, we remind the notion of the (n, ε) special convex combinations, (see [4],[6],[8]) which is one of the main tools, used in the sequel.

Definition 1.7. Let $x = \sum_{k \in F} c_k e_k$ be a vector in c_{00} . Then x is said to be a (n, ε) basic special convex combination (or a (n, ε) basic s.c.c.) if:

- (i) $F \in \mathcal{S}_n, c_k \geq 0$, for $k \in F$ and $\sum_{k \in F} c_k = 1$.
- (ii) For any $G \subset F, G \in \mathcal{S}_{n-1}$, we have that $\sum_{k \in G} c_k < \varepsilon$.

The proof of the next proposition can be found in [8], Chapter 2, Proposition 2.3.

Proposition 1.8. For any M infinite subset of the naturals, any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $F \subset M, \{c_k\}_{k \in F}$, such that $x = \sum_{k \in F} c_k e_k$ is a (n, ε) basic s.c.c.

Definition 1.9. Let $x_1 < \dots < x_m$ be vectors in c_{00} and $\psi(k) = \min \text{supp } x_k$, for $k = 1, \dots, m$. Then $x = \sum_{k=1}^m c_k x_k$ is said to be a (n, ε) special convex combination (or (n, ε) s.c.c.), if $\sum_{k=1}^m c_k e_{\psi(k)}$ is a (n, ε) basic s.c.c.

2. THE BASIC INEQUALITY

In this section we prove the basic inequality for block sequences in $\mathfrak{X}_{\text{ISP}}$, with the auxiliary space actually being Tsirelson space. This will allow us to evaluate the norm of (n, ε) special convex combinations and it is critical throughout the rest of the paper.

Definition 2.1. Let $f \in W$ be a functional of type I_α or I_β , of weight $w(f) = n$, $f = \frac{1}{2^n} \sum_{j=1}^d f_j$. Then, by definition, there exist $F_1 < \dots < F_p$ successive intervals of the naturals such that:

- (i) $\cup_{i=1}^p F_i = \{1, \dots, d\}$
- (ii) $\{\min \text{supp } f_j : j \in F_i\} \in \mathcal{S}_{n-1}$, for $i = 1, \dots, p$
- (iii) $\{\min \text{supp } f_{\min F_i} : i = 1, \dots, p\} \in \mathcal{S}_1$

Set $g_i = \frac{1}{2^{n-1}} \sum_{j \in F_i} f_j$, for $i = 1, \dots, p$. We call $\{g_i\}_{i=1}^p$ a Tsirelson analysis of f .

Remark 2.2. If $f \in W$ is a functional of type I_α or I_β and $\{f_i\}_{i=1}^p$ is a Tsirelson analysis of f , then $f_i \in W$, $\{f_i\}_{i=1}^p$ is \mathcal{S}_1 -admissible and $f = \frac{1}{2} \sum_{i=1}^p f_i$, although $\{f_i\}_{i=1}^p$ may not be a very fast growing sequence of α -averages or β -averages. Moreover, if $w(f) > 1$, then f_i is of the same type as f and $w(f_i) = w(f) - 1$ for $i = 1, \dots, p$.

The tree analysis of a functional $f \in W$. A key ingredient for evaluating the norm of vectors in $\mathfrak{X}_{\text{ISP}}$ is the analysis of the elements f of the norming set W . This is similar to the corresponding concept that has occurred in almost all previous HI and related constructions (i.e. [3], [4], [7], [8]). Next we briefly describe the tree analysis in our context.

For any functional $f \in W$ we associate a family $\{f_\lambda\}_{\lambda \in \Lambda}$, where Λ is a finite tree which is inductively defined as follows.

Set $f_\emptyset = f$, where \emptyset denotes the root of the tree to be constructed. If f is of type 0, then the tree analysis of f is $\{f_\emptyset\}$. Otherwise, suppose that the nodes of the tree and the corresponding functionals have been chosen up to a height p and let λ be a node of height $|\lambda| = p$. If f_λ is of type 0, then don't extend any further and λ is a maximal node of the tree.

If f_λ is of type I_α or I_β , set the immediate successors of λ to be the elements of the Tsirelson analysis of f_λ .

If f_λ is of type II, $f = \frac{1}{2} \sum_{j=1}^d f_j$, set the immediate successors of λ to be the $\{f_j\}_{j=1}^d$.

If f_λ is a convex combination, which includes α -averages and β -averages, $f_\lambda = \sum_{j=1}^d c_j f_j$, set the immediate successors of λ to be the $\{f_j\}_{j=1}^d$.

By Remark 1.3 it follows that the inductive construction ends in finitely many steps and that the tree Λ is finite.

Remark 2.3. Let $f \in W$ and $\{f_\lambda\}_{\lambda \in \Lambda}$ be a tree analysis of f . Then for any $\lambda \in \Lambda$ not a maximal node, such that f_λ is not a convex combination, we have that $f_\lambda = \frac{1}{2} \sum_{\mu \in \text{succ}(\lambda)} f_\mu$, where $\{f_\mu\}_{\mu \in \text{succ}(\lambda)}$ are \mathcal{S}_1 -admissible and by $\text{succ}(\lambda)$ we denote the immediate successors of λ in Λ .

Remark 2.4. In a simpler manner, for any $f \in W'_T$ (see Remark 1.4 (ii)), the tree analysis of f is defined.

Proposition 2.5. Let $x = \sum_{k \in F} c_k e_k$ be a (n, ε) basic s.c.c. and $G \subset F$. Then the following holds.

$$\left\| \sum_{k \in G} c_k e_k \right\|_T \leq \frac{1}{2^n} \sum_{k \in G} c_k + \varepsilon$$

Proof. Let $f \in W'_T$. We may assume that $\text{supp } f \subset G$. Set $G_1 = \{k \in \text{supp } f : |f(e_k)| \leq \frac{1}{2^n}\}$, $G_2 = \text{supp } f \setminus G_1$. Then clearly $|G_1 f(\sum_{k \in G} c_k e_k)| \leq \frac{1}{2^n} \sum_{k \in G} c_k$.

We will show by induction that $G_2 \in \mathcal{S}_{n-1}$. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a tree analysis of $G_2 f$. Then it is easy to see that $h(\Lambda) \leq n-1$. For λ a maximal node in Λ , we have that $\text{supp } f_\lambda \in \mathcal{S}_0$. Assume that for any $\lambda \in \Lambda$, $|\lambda| = k > 0$ we have that $\text{supp } f_\lambda \in \mathcal{S}_{n-1-k}$ and let $\lambda \in \Lambda$, such that $|\lambda| = k-1$. Then $f_\lambda = \frac{1}{2} \sum_{j=1}^d f_{\lambda_j}$, where $|\lambda_j| = k$, $\text{supp } f_{\lambda_j} \in \mathcal{S}_{n-k-1}$ for $j = 1, \dots, d$ and $\{\min \text{supp } f_{\lambda_j} : j = 1, \dots, d\} \in \mathcal{S}_1$. Then $\text{supp } f_\lambda = \cup_{j=1}^d \text{supp } f_{\lambda_j} \in \mathcal{S}_{n-1-(k-1)}$.

The induction is complete and it follows that $G_2 = \text{supp } G_2 f \in \mathcal{S}_{n-1}$ and therefore $G_2 f(\sum_{k \in G} c_k e_k) \leq \sum_{k \in G_2} c_k < \varepsilon$. Hence, $|f(\sum_{k \in G} c_k e_k)| < \frac{1}{2^n} \sum_{k \in G} c_k + \varepsilon$. \square

Proposition 2.6 (Basic Inequality). Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$ such that $\|x_k\| \leq 1$, for all k and let $f \in W$. Set $\phi(k) = \max \text{supp } x_k$, for all k . Then there exists $g \in W_{(T, \|\cdot\|)}$ (see Remark 1.6) such that $2g(e_{\phi(k)}) \geq f(x_k)$, for all k .

Proof. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a tree analysis of f . We will inductively construct $\{g_\lambda\}_{\lambda \in \Lambda}$ such that for any $\lambda \in \Lambda$ the following are satisfied.

- (i) $g_\lambda \in W_{(T, \|\cdot\|)}$ and $2g_\lambda(e_{\phi(k)}) \geq f_\lambda(x_k)$, for any k .
- (ii) $\text{supp } g_\lambda \subset \{\phi(k) : \text{ran } f_\lambda \cap \text{ran } x_k \neq \emptyset\}$

For $\lambda \in \Lambda$ a maximal node, if there exists k such that $\text{ran } f_\lambda \cap \text{ran } x_k \neq \emptyset$, set $g_\lambda = e_{\phi(k)}^*$. Otherwise set $g_\lambda = 0$.

Let $\lambda \in \Lambda$ be a non-maximal node, and suppose that $\{g_\mu\}_{\mu > \lambda}$ have been chosen. We distinguish two cases.

Case 1: f_λ is a convex combination (i.e. f_λ is not of type 0, I_α , I_β , or II).

If $f_\lambda = \sum_{\mu \in \text{succ}(\lambda)} c_\mu f_\mu$, set $g_\lambda = \sum_{\mu \in \text{succ}(\lambda)} c_\mu g_\mu$.

Case 2: f_λ is not a convex combination.

If $f_\lambda = \frac{1}{2} \sum_{j=1}^d f_{\mu_j}$, where $\text{succ}(\lambda) = \{\mu_j\}_{j=1}^d$ such that $f_{\mu_1} < \dots < f_{\mu_d}$, set

$$\begin{aligned} G_\lambda &= \{k : \text{ran } f_\lambda \cap \text{ran } x_k \neq \emptyset\} \\ G_1 &= \{k \in G_\lambda : \text{there exists at most one } j \text{ with } \text{ran } f_{\mu_j} \cap \text{ran } x_k \neq \emptyset\} \\ G_2 &= \{k \in G_\lambda : \text{there exist at least two } j \text{ with } \text{ran } f_{\mu_j} \cap \text{ran } x_k \neq \emptyset\} \\ I_j &= \{k \in G_1 : \text{ran } x_k \cap \text{ran } f_{\mu_j} \neq \emptyset\} \quad \text{for } j = 1, \dots, d. \end{aligned}$$

Observe that $\#G_2 \leq d - 1$.

For $j = 1, \dots, d$ set $g'_j = g_{\mu_j}|_{\phi(I_j)}$ and for $k \in G_2$ set $g_k = e_{\phi(k)}^*$. It is easy to check that if we set $g_\lambda = \frac{1}{2}(\sum_{j=1}^d g'_j + \sum_{k \in G_2} g_k)$, then g_λ is the desired functional.

The induction is complete. Set $g = g_\emptyset$

□

Remark 2.7. In the previous constructions (see [3], [4], [7], [8]), the basic inequality is used for estimating the norm of linear combinations of the basis as well as to determine whether particular sequences are RIS. In the present paper the basic inequality has a weaker role, namely only to estimate the norm of the (n, ε) s.c.c. In order to determine if a sequence is RIS, different techniques will be deployed in the next sections.

Corollary 2.8. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$ such that $\|x_k\| \leq 1$, $\{c_k\}_k \subset \mathbb{R}$ and $\phi(k) = \text{maxsupp } x_k$ for all k . Then:

$$\left\| \sum_k c_k x_k \right\| \leq 6 \left\| \sum_k c_k e_{\phi(k)} \right\|_T$$

Proof. Let $f \in W$. Apply the basic inequality and take $g \in W_{(T, \|\cdot\|)}$, such that if $\phi(k) = \text{maxsupp } x_k$ and $y_k = \text{sgn}(c_k)x_k$ for all k , we have that $2g(e_{\phi(k)}) \geq f(y_k)$, for any k . It follows that

$$2g\left(\sum_k |c_k| e_{\phi(k)}\right) \geq f\left(\sum_k c_k x_k\right).$$

Therefore, applying (1), we get

$$\left\| \sum_k c_k x_k \right\| \leq 2 \left\| \sum_k |c_k| e_{\phi(k)} \right\| = 2 \left\| \sum_k c_k e_{\phi(k)} \right\| \leq 2 \cdot 3 \left\| \sum_k c_k e_{\phi(k)} \right\|_T$$

□

Corollary 2.9. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| \leq 1$, for $k = 1, \dots, m$. If $F \subset \{1, \dots, m\}$, then

$$\left\| \sum_{k \in F} c_k x_k \right\| \leq \frac{6}{2^n} \sum_{k \in F} c_k + 12\varepsilon.$$

In particular, we have that $\|x\| \leq \frac{6}{2^n} + 12\varepsilon$.

Proof. Set $\phi(k) = \max \text{supp } x_k, \psi(k) = \min \text{supp } x_k$. Corollary 2.8 yields that $\|\sum_{k \in F} c_k x_k\| \leq 6 \|\sum_{k \in F} c_k e_{\phi(k)}\|_T$.

Since, according to the assumption, $\sum_{k \in F} c_k e_{\psi(k)}$ is a (n, ε) basic s.c.c., it easily follows that $\sum_{k \in F} c_k e_{\phi(k)}$ is a $(n, 2\varepsilon)$ basic s.c.c.

By Proposition 2.5 the result follows. \square

Corollary 2.10. The basis of $\mathfrak{X}_{\text{ISP}}$ is shrinking.

Proof. Suppose that it is not. Then there exist $x^* \in \mathfrak{X}_{\text{ISP}}^*, \|x^*\| = 1$, a normalized block sequence $\{x_k\}_{k \in \mathbb{N}}$ in $\mathfrak{X}_{\text{ISP}}$ and $\delta > 0$, such that $x^*(x_k) > \delta$, for all $k \in \mathbb{N}$.

Choose $n \in \mathbb{N}$, such that $\frac{1}{2^n} < \frac{\delta}{12}$ and $\varepsilon > 0$, such that $\varepsilon < \frac{\delta}{24}$. By Proposition 1.8 there exists F a subset of \mathbb{N} , such that $x = \sum_{k \in F} c_k x_k$ is a (n, ε) s.c.c.

By Corollary 2.9 we have that $\delta > \|x\| \geq x^*(x) > \delta$. A contradiction, which completes the proof. \square

Proposition 2.11. The basis of $\mathfrak{X}_{\text{ISP}}$ is boundedly complete.

Proof. Assume that it is not. Then there exist $\varepsilon > 0$ and $\{x_k\}_{k \in \mathbb{N}}$ a block sequence in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| > \varepsilon$ and $\|\sum_{k=\ell}^{\ell+m} x_k\| \leq 1$, for all $\ell, m \in \mathbb{N}$.

Choose k_0 such that $d = \min \text{supp } x_{k_0} > \frac{2}{\varepsilon}$. Set $F_1 = \{k_0\}$ and inductively choose F_1, \dots, F_d , intervals of the naturals such that

- (i) $\max F_j + 1 = \min F_{j+1}$, for $j < d$ and
- (ii) $\#F_j > \max\{\#F_{j-1}, (\max \text{supp } x_{\max F_{j-1}})^2\}$, for $1 < j \leq d$.

Then, if we set $y_j = \sum_{k \in F_j} x_k$, we have that $\|\sum_{j=1}^d y_j\| \leq 1$.

On the other hand, notice that for $j = 1, \dots, d$, there exists α_j an α -average in W , such that

- (i) $\text{ran } \alpha_j \subset \text{ran } y_j$, therefore $\{\alpha_j\}_{j=1}^d$ is \mathcal{S}_1 -admissible.
- (ii) $s(\alpha_j) = \#F_j$, therefore $\{\alpha_j\}_{j=1}^d$ is very fast growing.
- (iii) $\alpha_j(y_j) > \varepsilon$

From the above it follows $f = \frac{1}{2} \sum_{j=1}^d \alpha_j$ is a functional of type I_α in W and $f(\sum_{j=1}^d y_j) > \frac{\varepsilon \cdot d}{2} > 1$. Since this cannot be the case, the proof is complete. \square

These last two results and a well known result due to R. C. James [17], allow us to conclude the following.

Corollary 2.12. The space $\mathfrak{X}_{\text{ISP}}$ is reflexive.

3. THE α, β INDICES

To each block sequence we will associate two indices related to α and β averages. In this section we will show that every normalized block sequence $\{x_n\}_n$ has a further normalized block sequence $\{y_n\}_n$ such that on it both

indices α and β are equal to zero. As we will show in the next section, this is sufficient, for a sequence to have a subsequence generating a c_0 spreading model.

Definition 3.1. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$ that satisfies the following. For any $n \in \mathbb{N}$, for any very fast growing sequence $\{\alpha_q\}_{q \in \mathbb{N}}$ of α -averages in W and for any $\{F_k\}_{k \in \mathbb{N}}$ increasing sequence of subsets of the naturals, such that $\{\alpha_q\}_{q \in F_k}$ is \mathcal{S}_n -admissible, the following holds. For any $\{x_{n_k}\}_{k \in \mathbb{N}}$ subsequence of $\{x_k\}_{k \in \mathbb{N}}$, we have that $\lim_k \sum_{q \in F_k} |\alpha_q(x_{n_k})| = 0$.

Then we say that the α -index of $\{x_k\}_{k \in \mathbb{N}}$ is zero and write $\alpha(\{x_k\}_k) = 0$. Otherwise we write $\alpha(\{x_k\}_k) > 0$.

Definition 3.2. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$ that satisfies the following. For any $n \in \mathbb{N}$, for any very fast growing sequence $\{\beta_q\}_{q \in \mathbb{N}}$ of β -averages in W and for any $\{F_k\}_{k \in \mathbb{N}}$ increasing sequence of subsets of the naturals, such that $\{\beta_q\}_{q \in F_k}$ is \mathcal{S}_n -admissible, the following holds. For any $\{x_{n_k}\}_{k \in \mathbb{N}}$ subsequence of $\{x_k\}_{k \in \mathbb{N}}$, we have that $\lim_k \sum_{q \in F_k} |\beta_q(x_{n_k})| = 0$.

Then we say that the β -index of $\{x_k\}_{k \in \mathbb{N}}$ is zero and write $\beta(\{x_k\}_k) = 0$. Otherwise we write $\beta(\{x_k\}_k) > 0$.

Proposition 3.3. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$. Then the following assertions are equivalent.

- (i) $\alpha(\{x_k\}_k) = 0$
- (ii) For any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$ there exists $k_j \in \mathbb{N}$ such that for any $k \geq k_j$, and for any $\{\alpha_q\}_{q=1}^d$ \mathcal{S}_j -admissible and very fast growing sequence of α -averages such that $s(\alpha_q) > j_0$, for $q = 1, \dots, d$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < \varepsilon$.

Proof. It is easy to prove that (i) follows from (ii), therefore we shall only prove the inverse. Suppose that (i) is true and (ii) is not.

Then there exists $\varepsilon > 0$ such that for any $j_0 \in \mathbb{N}$ there exists $j \geq j_0$, such that for any $k_0 \in \mathbb{N}$, there exists $k \geq k_0$ and $\{\alpha_q\}_{q=1}^d$ a \mathcal{S}_j -admissible and very fast growing sequence of α -averages with $s(\alpha_q) > j_0$, for $q = 1, \dots, d$, such that $\sum_{q=1}^d |\alpha_q(x_k)| \geq \varepsilon$.

We will inductively choose a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ and $\{\alpha^i\}_{i \in \mathbb{N}}$ a very fast growing sequence of α -averages, such that $|\alpha^i(x_{n_i})| > \frac{\varepsilon}{2}$, for any i . This evidently yields a contradiction.

For $j_0 = 1$, there exists $j_1 \geq 1$, such that there exists a subsequence $\{x_{k_j}\}_{j \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$, a sequence $\{\alpha_q\}_{q \in \mathbb{N}}$ of α -averages with $s(\alpha_q) > 1$ for all $q \in \mathbb{N}$ and $\{F_j\}_{j \in \mathbb{N}}$ a sequence of increasing intervals of the naturals, such that:

- (i) $\{\alpha_q\}_{q \in F_j}$ is very fast growing and \mathcal{S}_{j_1} -admissible.
- (ii) $\sum_{q \in F_j} |\alpha_q(x_{k_j})| \geq \varepsilon$.
- (iii) If $F'_j = F_j \setminus \{\min F_j\}$, then $\{\alpha_q\}_{q \in \cup_j F'_j}$ is very fast growing.

Since $\alpha(\{x_k\}_k) = 0$, we have that $\lim_j \sum_{q \in F'_j} |\alpha_q(x_{k_j})| = 0$. Choose j such that $|\alpha_{\min F_j}(x_{k_j})| > \frac{\varepsilon}{2}$ and set $n_1 = k_j, \alpha^1 = \alpha_{\min F_j}$.

Suppose that we have chosen $n_1 < \dots < n_p$ and $\{a^i\}_{i=1}^p$ a very fast growing sequence of α -averages, such that $|\alpha^i(x_{n_i})| > \frac{\varepsilon}{2}$, for $i = 1, \dots, p$.

Set $j_0 = \max\{s(\alpha^p), (\max \supp \alpha^p)^2\}$ and repeat the first inductive step to find an α -average α with $s(\alpha) > j_0$ and $x_k > x_{n_p}, x_k > \alpha^p$, such that $|\alpha(x_k)| \geq \frac{\varepsilon}{2}$. Set $x_{n_{p+1}} = x_k$ and $\alpha^{p+1} = \alpha|_{\text{ran } x_k}$. The inductive construction is complete and so is the proof. \square

The proof of the next proposition is identical to the proof of the previous one.

Proposition 3.4. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$. Then the following assertions are equivalent.

- (i) $\beta(\{x_k\}_k) = 0$
- (ii) For any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for any $j \geq j_0$ there exists $k_j \in \mathbb{N}$ such that for any $k \geq k_j$, and for any $\{\beta_q\}_{q=1}^d$ \mathcal{S}_j -admissible and very fast growing sequence of β -averages such that $s(\beta_q) > j_0$, for $q = 1, \dots, d$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < \varepsilon$.

Proposition 3.5. Let $\{x_k\}_{k \in \mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}_{\text{ISP}}$, such that either $\alpha(\{x_k\}_k) > 0$, or $\beta(\{x_k\}_k) > 0$. Then there exists $c > 0$ and a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$, that generates an ℓ_1^n spreading model, with a lower constant $\frac{c}{2^n}$, for all $n \in \mathbb{N}$.

In particular, for any $k_0, n \in \mathbb{N}, \varepsilon > 0$, there exists F a finite subset of \mathbb{N} with $\min F \geq k_0$ and $\{c_k\}_{k \in F}$, such that $x = \sum_{k \in F} c_k x_{n_k}$ is a (n, ε) s.c.c. and $\|x\| > \frac{c}{2^n}$.

Proof. Assume that $\alpha(\{x_k\}_k) > 0$. Then there exist $\ell \in \mathbb{N}, \varepsilon > 0, \{\alpha_q\}_{q \in \mathbb{N}}$ a very fast growing sequence of α -averages, $\{F_k\}_{k \in \mathbb{N}}$ increasing subsets of the naturals such that $\{\alpha_q\}_{q \in F_k}$ is \mathcal{S}_ℓ -admissible for all $k \in \mathbb{N}$ and $\{x_{n_k}\}_{k \in \mathbb{N}}$ a subsequence of $\{x_k\}_{k \in \mathbb{N}}$, such that $\sum_{q \in F_k} |\alpha_q(x_{n_k})| > \varepsilon$, for all $k \in \mathbb{N}$. Pass, if necessary, to a subsequence, again denoted by $\{x_{n_k}\}_{k \in \mathbb{N}}$, generating some spreading model.

By changing the signs and restricting the ranges of the α_q , we may assume that $\sum_{q \in F_k} \alpha_q(x_{n_k}) > \varepsilon$, for all $k \in \mathbb{N}$ and $\text{ran } \alpha_q \subset \text{ran } x_{n_k}$ for any $q \in F_k$ and $k \in \mathbb{N}$. Set $c = \frac{\varepsilon}{2^\ell}$.

Let $k_0, n \in \mathbb{N}, \varepsilon > 0$. By Proposition 1.8 there exists F a finite subset of $\{n_k : k \geq k_0\}$ and $\{c_k\}_{k \in F}$, such that $x = \sum_{k \in F} c_k x_{n_k}$ is a (n, ε) s.c.c.

Set $f = \frac{1}{2^{\ell+n}} \sum_{k \in F} \sum_{q \in F_{n_k}} \alpha_q$. Then f is a functional of type I_α in W and $f(x) > \frac{\varepsilon}{2^{\ell+n}} = \frac{c}{2^n}$.

Arguing in the same way, for any $n \in \mathbb{N}$, for any $F \in \mathcal{S}_n$, for any $\{c_k\}_{k \in F} \subset \mathbb{R}$, we have that $\|\sum_{k \in F} c_k x_{n_k}\| > \frac{c}{2^n} \sum_{k \in F} |c_k|$.

The proof is exactly the same if $\beta(\{x_k\}_k) > 0$. \square

Block sequences with α -index zero. In this subsection we show that seminormalized sequences $\{x_k\}_{k \in \mathbb{N}}$ with $x_k = 2^{n_k} y_k$, with $y_k (n_k, \varepsilon_k)$ s.c.c. have α -index zero. Also we introduce the α -RIS sequences and we prove that the aforementioned sequences have α -RIS subsequences.

Lemma 3.6. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| \leq 1$, for $k = 1, \dots, m$. Let also α be an α -average and set $G_\alpha = \{k : \text{ran } \alpha \cap \text{ran } x_k \neq \emptyset\}$. Then the following holds.

$$|\alpha(x)| < \min \left\{ \frac{1}{s(\alpha)} \sum_{k \in G_\alpha} c_k, \frac{6}{2^n s(\alpha)} \sum_{k \in G_\alpha} c_k + 12\varepsilon \right\} + 2 \max\{c_k : k \in G_\alpha\}$$

Proof. If $\alpha = \frac{1}{p} \sum_{j=1}^d f_j$. Set

$$\begin{aligned} E_1 &= \{k \in G_\alpha : \text{there exists at most one } j \text{ with } \text{ran } f_j \cap \text{ran } x_k \neq \emptyset\} \\ E_2 &= \{1, \dots, m\} \setminus E_1 \\ J_k &= \{j : \text{ran } f_j \cap \text{ran } x_k \neq \emptyset\} \quad \text{for } k \in E_2. \end{aligned}$$

Then it is easy to see that

$$(2) \quad |\alpha(\sum_{k \in E_1} c_k x_k)| \leq \frac{1}{p} \sum_{k \in G_\alpha} c_k$$

Moreover

$$(3) \quad |\alpha(\sum_{k \in E_2} c_k x_k)| < 2 \max\{c_k : k \in G_\alpha\}$$

To see this, notice that

$$|\alpha(\sum_{k \in E_2} c_k x_k)| \leq \frac{1}{p} \sum_{k \in E_2} c_k \left(\sum_{j \in J_k} |f_j(x_k)| \right) < \max\{c_k : k \in G_\alpha\} \frac{2p}{p}$$

Set $J = \{j : \text{there exists } k \in E_1 \text{ such that } \text{ran } f_j \cap \text{ran } x_k \neq \emptyset\}$ and for $j \in J$ set $G_j = \{k \in E_1 : \text{ran } f_j \cap \text{ran } x_k \neq \emptyset\}$. Then the G_j are pairwise disjoint and $\cup_{j \in J} G_j = E_1$.

For $j \in J$, Corollary 2.9 yields that $|f_j(\sum_{k \in G_j} c_k x_k)| \leq \frac{6}{2^n} \sum_{k \in G_j} c_k + 12\varepsilon$.

Therefore

$$(4) \quad |\alpha(\sum_{k \in E_1} c_k x_k)| \leq \frac{1}{p} \sum_{j \in J} |f_j(\sum_{k \in G_j} c_k x_k)| \leq \frac{6}{2^n p} \sum_{k \in G_\alpha} c_k + 12\varepsilon$$

Then (2) and (4) yield the following.

$$(5) \quad |\alpha(\sum_{k \in E_1} c_k x_k)| \leq \min \left\{ \frac{1}{s(\alpha)} \sum_{k \in G_\alpha} c_k, \frac{6}{2^n s(\alpha)} \sum_{k \in G_\alpha} c_k + 12\varepsilon \right\}$$

By summing up (3) and (5) the result follows. \square

Lemma 3.7. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| \leq 1$, for $k = 1, \dots, m$. Let also $\{a_q\}_{q=1}^d$ be a very fast growing and \mathcal{S}_j -admissible sequence of α -averages, with $j < n$. Then the following holds.

$$2^n \sum_{q=1}^d |\alpha_q(x)| < \frac{6}{s(\alpha_1)} + \frac{2 \cdot 2^n}{\min \text{supp } x} + 18 \cdot 2^n \varepsilon + 2^{2n} \varepsilon$$

Proof. Set $q_1 = \min\{q : \text{ran } \alpha_q \cap \text{ran } x \neq \emptyset\}$. For convenience assume that $q_1 = 1$. Then by Lemma 3.6 we have that

$$(6) \quad 2^n |\alpha_1(x)| < \frac{6}{s(\alpha_1)} + 12 \cdot 2^n \varepsilon + 2 \cdot 2^n \varepsilon$$

Set

$$\begin{aligned} J_1 &= \{q > 1 : \text{there exists at most one } k \text{ such that } \text{ran } \alpha_q \cap \text{ran } x_k \neq \emptyset\} \\ J_2 &= \{q > 1 : q \notin J_1\} \\ G^q &= \{k : \text{ran } \alpha_q \cap \text{ran } x_k \neq \emptyset\} \quad \text{for } q > 1. \\ G_1 &= \{k : \text{there exists } q \in J_1 \text{ with } \text{ran } \alpha_q \cap \text{ran } x_k \neq \emptyset\} \end{aligned}$$

Then $\{\min \text{supp } x_k : k \in G_1\} \in \mathcal{S}_j$, hence $\sum_{k \in G_1} c_k < \varepsilon$.

It is easy to check that

$$(7) \quad 2^n \sum_{q \in J_1} |\alpha_q(x)| \leq 2^j \cdot 2^n \left\| \sum_{k \in G_1} c_k x_k \right\| < 2^{2n} \varepsilon$$

For $q \in J_2$, Lemma 3.6 yields that

$$\begin{aligned} 2^n |\alpha_q(x)| &< \frac{2^n}{s(\alpha_q)} \sum_{k \in G^q} c_k + 2 \cdot 2^n \max\{c_k : k \in G^q\} \\ &< \frac{2^n}{\min \text{supp } x} \sum_{k \in G^q} c_k + 2 \cdot 2^n c_{k_q} \end{aligned}$$

where $k_q \in G^q$, such that $c_{k_q} = \max\{c_k : k \in G^q\}$.

Then $\{\min \text{supp } x_{k_q} : q \in J_2\} \in \mathcal{S}_j$. By the above we conclude that

$$(8) \quad 2^n \sum_{q \in J_2} |\alpha_q(x)| < \frac{2 \cdot 2^n}{\min \text{supp } x} + 4 \cdot 2^n \varepsilon$$

Summing up (6), (7) and (8), the desired result follows. \square

Proposition 3.8. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$ such that $x_k = 2^{n_k} \sum_{i \in F_k} c_i^k y_i^k$ satisfying the following:

- (i) $\{n_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of naturals.
- (ii) $\sum_{i \in F_k} c_i^k y_i^k$ is a (n_k, ε_k) s.c.c. such that $\|y_i^k\| \leq 1$ for all $i \in F_k$, for all $k \in \mathbb{N}$.
- (iii) $\min \text{supp } x_k > 4 \cdot 2^{2n_k}$ and $\varepsilon_k < (40 \cdot 2^{3n_k})^{-1}$, for all $k \in \mathbb{N}$.

Then $\alpha(\{x_k\}_k) = 0$.

Proof. We shall make use of Proposition 3.3. Let $\varepsilon > 0$ and choose $j_0 \in \mathbb{N}$ such that $\frac{6}{j_0} < \frac{\varepsilon}{2}$. For $j \geq j_0$, choose k_j , such that $n_{k_j} > j$ and $\frac{1}{2^{n_{k_j}}} < \frac{\varepsilon}{2}$. For $k \geq k_j$, Lemma 3.7 yields that if $\{\alpha_q\}_{q=1}^d$ is a very fast growing and \mathcal{S}_j -admissible sequence of α -averages and $s(\alpha_q) > j_0$, for $q = 1, \dots, d$, we have that

$$\sum_{q=1}^d |\alpha_q(x_k)| < \frac{6}{j_0} + \frac{1}{2^{n_k}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Proposition 3.9. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| \leq 1$, for $k = 1, \dots, m$, $\min \text{supp } x > 4 \cdot 2^{2n}$ and $\varepsilon < (40 \cdot 2^{3n})^{-1}$. Then for any $f \in W$ functional of type I_α , such that $w(f) = j < n$, we have that $2^n |f(x)| < \frac{7}{2^j}$.

Proof. Let $f = \frac{1}{2^j} \sum_{q=1}^d \alpha_q$ be a functional of type I_α with weight $w(f) = j < n$. Then Lemma 3.7 yields that

$$2^n |f(x)| \leq \frac{1}{2^j} \left(2^n \sum_{q=1}^d |\alpha_q(x)| \right) < \frac{1}{2^j} \left(\frac{6}{s(\alpha_1)} + \frac{2 \cdot 2^n}{\min \text{supp } x} + 18 \cdot 2^n \varepsilon + 2^{2n} \varepsilon \right) \leq \frac{7}{2^j}$$

□

Definition 3.10. A block sequence $\{x_k\}_k$ is said to be a $(C, \{n_k\}_k)$ α -rapidly increasing sequence (or $(C, \{n_k\}_k)$ α -RIS), for a positive constant $C \geq 1$ and a strictly increasing sequence of naturals $\{n_k\}_k$, if $\|x_k\| \leq C$ for all k and the following conditions are satisfied.

- (i) For any k , for any functional f of type I_α of weight $w(f) = j < n_k$ we have that $|f(x_k)| < \frac{C}{2^j}$
- (ii) For any k we have that $\frac{1}{2^{n_{k+1}}} \max \text{supp } x_k < \frac{1}{2^{n_k}}$

Remark 3.11. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$, such that there exist a positive constance C and $\{n_k\}_{k \in \mathbb{N}}$ strictly increasing naturals, such that $\|x_k\| \leq C$ for all k and condition (i) from Definition 3.10 is satisfied. Then passing, if necessary, to a subsequence, $\{x_k\}_{k \in \mathbb{N}}$ is $(C, \{n_k\}_{k \in \mathbb{N}})$ α -RIS.

Proposition 3.12. Let $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence in $\mathfrak{X}_{\text{ISP}}$ with $x_k = 2^{n_k} \sum_{i \in F_k} c_i^k y_i^k$ satisfying the following:

- (i) $\{n_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of naturals.
- (ii) $\sum_{i \in F_k} c_i^k y_i^k$ is a (n_k, ε_k) s.c.c. such that $\|y_i^k\| \leq 1$ for all $i \in F_k$, for all $k \in \mathbb{N}$.
- (iii) $\min \text{supp } x_k > 4 \cdot 2^{2n_k}$ and $\varepsilon_k < (40 \cdot 2^{3n_k})^{-1}$, for all $k \in \mathbb{N}$.

Then passing, if necessary, to a subsequence, $\{x_k\}_{k \in \mathbb{N}}$ is $(7, \{n_k\}_k)$ α -RIS.

Proof. Corollary 2.9 yields that $\|x_k\| < 7$ while Proposition 3.9 yields that (i) from Definition 3.10 is satisfied. By Remark 3.11 the result follows. □

Block sequences with β -index zero. In this subsection we first prove that every increasing seminormalized (n, ε_n) s.c.c. built on an α -RIS block sequence has β -index zero. This yields that every block sequence has a further block sequence with both α, β indices equal to zero. We start with the following technical lemma. Its meaning becomes more transparent in the following Corollary 3.14 and Lemmas 3.15, 3.16.

Notation. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. such that $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α -RIS with $2^{2n} < n_1$. Let also $f = \frac{1}{2} \sum_{j=1}^d f_j$ be a type II functional. Set

$$\begin{aligned} I_0 &= \{j : n \leq w(f_j) < 2^{2n}\} \\ I_1 &= \{j : w(f_j) < n\} \\ I_2 &= \{j : 2^{2n} \leq w(f_j) < n_1\} \\ J_k &= \{j : n_k \leq w(f_j) < n_{k+1}\}, \text{ for } k < m \text{ and } J_m = \{j : n_m \leq w(f_j)\} \end{aligned}$$

Under the above notation the following lemma holds.

Lemma 3.13. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}, n \geq 2$ such that the following are satisfied:

- (i) $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α -RIS with $2^{2n} < n_1$.
- (ii) $\min \text{supp } x > 4C \cdot 2^{2n}$ and $\varepsilon < (40C2^{3n})^{-1}$.

Let also $f = \frac{1}{2} \sum_{j=1}^d f_j$ be a functional of type II. Then there exists $F_f \subset \{x_k : \text{ran } f \cap \text{ran } x_k \neq \emptyset\}$ with $\{\min \text{supp } x_k : k \in F_f\} \in \mathcal{S}_2$ such that

$$\begin{aligned} 2^n |f(x)| &< 7C \#I_0 + \frac{C}{2} \left(\sum_{k=2}^m \sum_{j \in J_k} \frac{2^{n_k}}{2^{w(f_j) + n_{k-1}}} + \sum_{k=1}^{m-1} \sum_{j \in J_k} \frac{2^n}{2^{w(f_j)}} \right. \\ &\quad \left. + \sum_{j \in I_1} \frac{7}{2^{w(f_j)}} + \sum_{j \in I_2} \frac{2^n}{2^{w(f_j)}} \right) + C2^n \sum_{k \in F_f} c_k \end{aligned}$$

Proof. Notice that $\{J_k\}_{k=1}^m$ are disjoint intervals of $\{1, \dots, d\}$ and that $g_k = \frac{1}{2} \sum_{j \in J_k} f_j \in W$, for $k = 1, \dots, m$.

Set $F_f = \{k : \text{ran } g_k \cap \text{ran } x_k \neq \emptyset\}$. It easily follows that $\{\min \text{supp } x_k : k \in F_f\} \in \mathcal{S}_2$ and that

$$(9) \quad \frac{2^n}{2} \sum_{k=1}^m \left| \sum_{j \in J_k} f_j(c_k x_k) \right| \leq C2^n \sum_{k \in F_f} c_k$$

Let $k_0 \leq m, j \in J_{k_0}$. Then

$$(10) \quad 2^n |f_j(\sum_{k < k_0} c_k x_k)| < C \frac{2^{n_{k_0}}}{2^{w(f_j) + n_{k_0-1}}} \text{ and } 2^n |f_j(\sum_{k > k_0} c_k x_k)| < C \frac{2^n}{2^{w(f_j)}}$$

Proposition 3.9 yields that for $j \in I_1$ we have that $2^n |f_j(x)| < \frac{7C}{2^{w(f_j)}}$ and hence

$$(11) \quad \frac{2^n}{2} \sum_{j \in I_1} |f_j(x)| < \frac{C}{2} \sum_{j \in I_1} \frac{7}{2^{w(f_j)}}$$

For $j \in I_2$ we have that $2^n |f_j(x)| < \frac{C2^n}{2^{w(f_j)}}$ and therefore

$$(12) \quad \frac{2^n}{2} \sum_{j \in I_2} |f_j(x)| < \frac{C}{2} \sum_{j \in I_2} \frac{2^n}{2^{w(f_j)}}$$

Corollary 2.9 yields that $2^n \|x\| < 7C$, and since I_0 is an interval, it follows that $\frac{1}{2} \sum_{j \in I_0} f_j \in W$. Therefore

$$(13) \quad \frac{2^n}{2} \left| \sum_{j \in I_0} f_j(x) \right| < 7C$$

Summing up (9) to (13) the desired result follows. \square

The next corollary will be useful in the next sections, when we define the notion of dependent sequences.

Corollary 3.14. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, $n \geq 3$ such that the following are satisfied:

- (i) $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α -RIS with $2^{2n} < n_1$.
- (ii) $\min \text{supp } x > 4C \cdot 2^{2n}$ and $\varepsilon < (40C2^{3n})^{-1}$.

Let also $f = \frac{1}{2} \sum_{j=1}^d f_j$ be a functional of type II with $\widehat{w}(f) \cap \{n, \dots, 2^{2n}\} = \emptyset$. Then

$$2^n |f(x)| < C \left(\frac{1}{2^n} + \frac{1}{2^{2n}} + \sum_{\{j: w(f_j) < n\}} \frac{4}{2^{w(f_j)}} + 2^n \varepsilon \right)$$

Proof. Apply Lemma 3.13. Then the following holds.

$$(14) \quad 2^n |f(x)| < \frac{C}{2} \left(\sum_{k=2}^m \sum_{j \in J_k} \frac{2^{n_k}}{2^{w(f_j) + n_{k-1}}} + \sum_{k=1}^{m-1} \sum_{j \in J_k} \frac{2^n}{2^{w(f_j)}} \right. \\ \left. + \sum_{j \in I_1} \frac{7}{2^{w(f_j)}} + \sum_{j \in I_2} \frac{2^n}{2^{w(f_j)}} \right) + C2^n \varepsilon$$

Notice the following.

$$(15) \quad \sum_{k=2}^m \sum_{j \in J_k} \frac{2^{n_k}}{2^{w(f_j) + n_{k-1}}} \leq \frac{1}{2^{n_1}} < \frac{1}{2^{2n}}$$

$$(16) \quad \sum_{j \in I_2} \frac{2^n}{2^{w(f_j)}} + \sum_{k=1}^{m-1} \sum_{j \in J_k} \frac{2^n}{2^{w(f_j)}} = 2^n \left(\sum_{\{j: w(f_j) \geq 2^{2n}\}} \frac{1}{2^{w(f_j)}} \right) \leq \frac{2}{2^n}$$

Applying (15) and (16) to (14) the result follows. \square

Lemma 3.15. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, $n \geq 2$ such that the following are satisfied:

- (i) $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α -RIS with $2^{2n} < n_1$.
- (ii) $\min \text{supp } x > 4C2^{2n}$ and $\varepsilon < (40C2^{3n})^{-1}$.

Let also β be a β -average. Then there exists $F_\beta \subset \{x_k : \text{ran } \beta \cap \text{ran } x_k \neq \emptyset\}$ with $\{\min \text{supp } x_k : k \in F_\beta\} \in \mathcal{S}_2$ such that

$$2^n |\beta(x)| < \frac{8C}{s(\beta)} + C2^n \sum_{k \in F_\beta} c_k$$

Proof. If $\beta = \frac{1}{p} \sum_{q=1}^p f_q$, then by definition the f_q are functionals of type II with disjoint weights $\widehat{w}(f_q)$.

For convenience, we may write $f_q = \frac{1}{2} \sum_{j \in G_q} f_j$, where the index sets $G_q, q = 1, \dots, p$ are pairwise disjoint. Notice that for $j_1, j_2 \in G, j_1 \neq j_2$ we have that $w(f_{j_1}) \neq w(f_{j_2})$.

By slightly modifying the previously used notation, set $G = \cup_{q=1}^p G_q$ and

$$\begin{aligned} I_0 &= \{j \in G : n \leq w(f_j) < 2^{2n}\} \\ I_1 &= \{j \in G : w(f_j) < n\} \\ I_2 &= \{j \in G : 2^{2n} \leq w(f_j) < n_1\} \\ J_k &= \{j \in G : n_k \leq w(f_j) < n_{k+1}\}, \text{ for } k < m \text{ and} \\ J_m &= \{j \in G : n_m \leq w(f_j)\} \end{aligned}$$

By Remark 1.1 there exists at most one $q_0 \leq d$, with $\widehat{w}(f_{q_0}) \cap \{n, \dots, 2^{2n}\} \neq \emptyset$ and if such a q_0 exists, then $\#\widehat{w}(f_{q_0}) \cap \{n, \dots, 2^{2n}\} \leq 1$.

Apply Lemma 3.13. Then for $q = 1, \dots, d$ there exists $F_q \subset \{x_k : \text{ran } \beta \cap \text{ran } x_k \neq \emptyset\}$ with $\{\min \text{supp } x_k : k \in F_q\} \in \mathcal{S}_2$ such that

$$\begin{aligned} (17) \quad 2^n |\beta(x)| &< \frac{7C}{p} + \frac{C}{2p} \left(\sum_{k=2}^m \sum_{j \in J_k} \frac{2^{n_k}}{2^{w(f_j) + n_{k-1}}} + \sum_{k=1}^{m-1} \sum_{j \in J_k} \frac{2^n}{2^{w(f_j)}} \right. \\ &\quad \left. + \sum_{j \in I_1} \frac{7}{2^{w(f_j)}} + \sum_{j \in I_2} \frac{2^n}{2^{w(f_j)}} \right) + \frac{1}{p} C 2^n \sum_{q=1}^p \sum_{k \in F_q} c_k \end{aligned}$$

Just as in the proof of Corollary 3.14, notice the following.

$$(18) \quad \sum_{k=2}^m \sum_{j \in J_k} \frac{2^{n_k}}{2^{w(f_j) + n_{k-1}}} < \frac{1}{2^{2n}}$$

$$(19) \quad \sum_{j \in I_2} \frac{2^n}{2^{w(f_j)}} + \sum_{k=1}^{m-1} \sum_{j \in J_k} \frac{2^n}{2^{w(f_j)}} \leq \frac{2}{2^n}$$

By the definition of the coding function σ we get

$$(20) \quad \sum_{j \in I_1} \frac{7}{2^{w(f_j)}} < \frac{7}{1000}$$

$$(21) \quad \frac{1}{p} C 2^n \sum_{q=1}^p \sum_{k \in F_q} c_k \leq C 2^n \max \left\{ \sum_{k \in F_q} c_k : q = 1, \dots, p \right\} = C 2^n \sum_{k \in F_{q_0}} c_k$$

for some $1 \leq q_0 \leq p$.

Set $F_\beta = F_{q_0}$ and apply (18) to (21) to (17) to derive the desired result. \square

Lemma 3.16. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{ISP}}$, $n \geq 4$ such that the following are satisfied:

- (i) $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α -RIS with $2^{2n} < n_1$.
- (ii) $\min \text{supp } x > 4C 2^{2n}$ and $\varepsilon < (40C 2^{3n})^{-1}$.

Let also $\{\beta_q\}_{q=1}^d$ be a very fast growing and \mathcal{S}_j -admissible sequence of β -averages with $j \leq n - 3$. Then we have that

$$2^n \sum_{q=1}^d |\beta_q(x)| < \sum_{q=1}^d \frac{8C}{s(\beta_q)} + 2C 2^n \varepsilon + C 2^{2n} \varepsilon$$

Proof. Set

$$J_1 = \{q : \text{there exists at most one } k \text{ such that } \text{ran } \beta_q \cap \text{ran } x_k \neq \emptyset\}$$

$$J_2 = \{1, \dots, d\} \setminus J_1$$

$$G_1 = \{k : \text{there exists } q \in J_1 \text{ with } \text{ran } \beta_q \cap \text{ran } x_k \neq \emptyset\}$$

Then $\{\min \text{supp } x_k : k \in G_1\} \in \mathcal{S}_{n-2}$ and it is easy to check that

$$(22) \quad 2^n \sum_{q \in J_1} |\beta_q(x)| \leq 2^j 2^n \left\| \sum_{k \in G_1} c_k x_k \right\| < C 2^{2n} \varepsilon$$

For $q \in J_2$, choose $F_q \subset \{1, \dots, m\}$ as in Lemma 3.15 and set $F = \cup_{q \in J_2} F_q$. Then $\{\min \text{supp } x_k : k \in F\} \in \mathcal{S}_{n-1}$, therefore $\sum_{q \in J_2} \sum_{k \in F_q} c_k < 2\varepsilon$.

Lemma 3.15 yields that

$$(23) \quad 2^n \sum_{q \in J_2} |\beta_q(x)| < \sum_{q \in J_2} \frac{8C}{s(\beta_q)} + 2C 2^n \varepsilon$$

Combining (22) and (23), the result follows. \square

Proposition 3.17. Let $\{y_i\}_{i \in \mathbb{N}}$ be $(C, \{n_i\}_i)$ α -RIS in $\mathfrak{X}_{\text{ISP}}$, $\{x_k\}_{k \in \mathbb{N}}$ be a block sequence of $\{y_i\}_{i \in \mathbb{N}}$, $x_k = 2^{m_k} \sum_{i \in F_k} c_i^k y_i$ satisfying the following:

- (i) $\{m_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of naturals.
- (ii) $\sum_{i \in F_k} c_i^k y_i$ is a (m_k, ε_k) s.c.c. with $2^{2m_k} < n_{\min F_k}$.

(iii) $\min \text{supp } x_k > 4C2^{2m_k}$ and $\varepsilon_k < (40C2^{3m_k})^{-1}$, for all $k \in \mathbb{N}$.

Then $\alpha(\{x_k\}_k) = 0$ as well as $\beta(\{x_k\}_k) = 0$.

Proof. Proposition 3.8 yields that $\alpha(\{x_k\}_k) = 0$. To prove that $\beta(\{x_k\}_k) = 0$, we shall make use of Proposition 3.4. Let $\varepsilon > 0$ and choose $j_0 \in \mathbb{N}$ such that

$$\frac{8C}{j_0} + \sum_{j > j_0} \frac{8C}{j^2} < \frac{\varepsilon}{2}$$

For $j \geq j_0$ choose k_j , such that $m_{k_j} \geq j + 3$ and $\frac{3}{40 \cdot 2^{m_{k_j}}} < \frac{\varepsilon}{2}$. For $k \geq k_j$, Lemma 3.16 yields that if $\{\beta_q\}_{q=1}^d$ is a very fast growing and \mathcal{S}_j -admissible sequence of β -averages and $s(\beta_q) > j_0$, for $q = 1, \dots, d$, we have that

$$\sum_{q=1}^d |\beta_q(x_k)| < \frac{8C}{j_0} + \sum_{j > j_0} \frac{8C}{j^2} + \frac{3}{40 \cdot 2^{m_k}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Corollary 3.18. Let $\{x_k\}_{k \in \mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}_{\text{ISP}}$. Then there exists a further normalized block sequence $\{y_k\}_{k \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$, such that $\alpha(\{y_k\}_k) = 0$ as well as $\beta(\{y_k\}_k) = 0$.

Proof. If $\alpha(\{x_k\}_k) = 0$ and $\beta(\{x_k\}_k) = 0$, then there is nothing to prove. Otherwise, if $\alpha(\{x_k\}_k) > 0$ or $\beta(\{x_k\}_k) > 0$, apply Proposition 3.5 to construct a seminormalized block sequence $\{z_k\}_{k \in \mathbb{N}}$, satisfying the assumption of Proposition 3.8. Then $\alpha(\{z_k\}_k) = 0$. Proposition 3.12 yields, that passing, if necessary, to a subsequence, we have that $\{z_k\}_{k \in \mathbb{N}}$ is $(7, \{n_k\}_k)$ α -RIS.

If $\beta(\{z_k\}_k) = 0$, set $y_k = \frac{1}{\|z_k\|} z_k$ and $\{y_k\}_{k \in \mathbb{N}}$ is the desired sequence.

Otherwise, if $\beta(\{z_k\}_k) > 0$, apply once more Proposition 3.5 to construct a seminormalized block sequence $\{w_k\}_{k \in \mathbb{N}}$, satisfying the assumption of Corollary 3.17. Set $y_k = \frac{1}{\|w_k\|} w_k$ and $\{y_k\}_{k \in \mathbb{N}}$ is the desired sequence.

□

4. c_0 SPREADING MODELS

This section is devoted to necessary conditions for a sequence $\{x_k\}_k$ to generate a c_0 spreading model. At the beginning a Ramsey type result is proved concerning type II functionals acting on a block sequence $\{x_k\}_k$ with $\beta(\{x_k\}_k) = 0$. Then conditions are provided for a finite sequence to be equivalent to the basis of ℓ_∞^n . This is critical for establishing the HI property and the properties of the operators in the space. Moreover it is shown that any block sequence $\{x_k\}_k$ with $\alpha(\{x_k\}_k) = 0$ and $\beta(\{x_k\}_k) = 0$ contains a subsequence generating a c_0 spreading model. Another critical property related to sequences generating c_0 spreading models is that increasing Schreier sums of them define α -RIS sequences.

A combinatorial result.

Definition 4.1. Let $x_1 < x_2 < x_3$ be vectors in $\mathfrak{X}_{\text{ISP}}$, $f = \frac{1}{2} \sum_{j=1}^d f_j$ be a functional of type II, such that $\text{supp } f \cap \text{ran } x_i \neq \emptyset$, for $i = 1, 2, 3$ and $j_0 = \min\{j : \text{ran } f_j \cap \text{ran } x_2 \neq \emptyset\}$. If $\text{ran } f_{j_0} \cap \text{ran } x_3 = \emptyset$, then we say that f separates x_1, x_2, x_3 .

Definition 4.2. Let $i, j \in \mathbb{N}$. If there exists $f \in W$ a functional of type II, such that $i, j \in \widehat{w}(f)$, then we say that i is compatible to j .

Lemma 4.3. Let $x_1 < x_2 < \dots < x_m$ be vectors in $\mathfrak{X}_{\text{ISP}}$, such that there exist $\varepsilon > 0$ and $\{f_k\}_{k=2}^{m-1}$ functionals of type II satisfying the following.

- (i) f_k separates x_1, x_k, x_m , for $k = 2, \dots, m-1$
- (ii) If $f_k = \frac{1}{2} \sum_{j=1}^{d_k} f_j^k$ and $j_k = \min\{j : \text{ran } f_j^k \cap \text{ran } x_k \neq \emptyset\}$, then $w(f_{j_k}^k)$ is not compatible to $w(f_{j_\ell}^\ell)$ for $k \neq \ell$.
- (iii) $|f_k(x_m)| > \varepsilon$ for $k = 2, \dots, m-1$

Then there exists a β -average β in W of size $s(\beta) = m-2$ such that $\beta(x_m) > \varepsilon$.

Proof. Set $g_k = \text{sgn}(f_k(x_m))f_k|_{\text{ran } x_m}$, for $k = 2, \dots, m-1$. Then g_k is a functional of type II in W . We will show that the g_k have disjoint weights $\widehat{w}(g_k)$.

Towards a contradiction, suppose that there exist $k \neq \ell$ and $i \in \widehat{w}(g_k) \cap \widehat{w}(g_\ell)$. By (i) and the way type II functionals are constructed, it follows that $f_k|_{[\min \text{supp } x_2, \dots, \min \text{supp } x_{m-1}]} = f_\ell|_{[\min \text{supp } x_2, \dots, \min \text{supp } x_{m-1}]}$. This contradicts (ii).

By the above, it follows that if we set $\beta = \frac{1}{m-2} \sum_{k=2}^{m-1} g_k$, then β is the desired β -average. \square

Lemma 4.4. Let $x_1 < x_2 < \dots < x_m$ be vectors in $\mathfrak{X}_{\text{ISP}}$, such that there exist $\varepsilon > 0$ and $\{f_k\}_{k=2}^{m-1}$ functionals of type II satisfying the following.

- (i) f_k separates x_1, x_k, x_m , for $k = 2, \dots, m-1$
- (ii) If $f_k = \frac{1}{2} \sum_{j=1}^{d_k} f_j^k$ and $j_k = \min\{j : \text{ran } f_j^k \cap \text{ran } x_k \neq \emptyset\}$, then $w(f_{j_k}^k) = w(f_{j_\ell}^\ell)$ for $k \neq \ell$.
- (iii) If $j'_k = \min\{j : \text{ran } f_j^k \cap \text{ran } x_m \neq \emptyset\}$, then $w(f_{j'_k}^k) \neq w(f_{j'_\ell}^\ell)$ for $k \neq \ell$.
- (iv) $|f_k(x_m)| > \varepsilon$ for $k = 2, \dots, m-1$

Then there exists a β -average β in W of size $s(\beta) = m-2$ such that $\beta(x_m) > \varepsilon$.

Proof. As before, set $g_k = \text{sgn}(f_k(x_m))f_k|_{\text{ran } x_m}$, for $k = 2, \dots, m-1$. Then g_k is a functional of type II in W . We will show that the g_k have disjoint weights $\widehat{w}(g_k)$.

Suppose that there exist $k \neq \ell$ and $i \in \widehat{w}(g_k) \cap \widehat{w}(g_\ell)$. By (i), (ii) and the way type II functionals are constructed, it follows that

$$f_k|_{[\min \text{supp } x_2, \dots, \min \text{supp } x_m]} = f_\ell|_{[\min \text{supp } x_2, \dots, \min \text{supp } x_m]}$$

This leaves us no choice, but to conclude that $w(f_{j'_k}^k) = w(f_{j'_\ell}^\ell)$, a contradiction.

It follows that if we set $\beta = \frac{1}{m-2} \sum_{k=2}^{m-1} g_k$, then β is the desired β -average. \square

Proposition 4.5. Let $\{x_k\}_{k \in \mathbb{N}}$ be a bounded block sequence in $\mathfrak{X}_{\text{ISP}}$, such that $\beta(\{x_k\}_k) = 0$. Then for any $\varepsilon > 0$, there exists M an infinite subset of the naturals, such that for any $k_1 < k_2 < k_3 \in M$, for any functional $f \in W$ of type II that separates $x_{k_1}, x_{k_2}, x_{k_3}$, we have that $|f(x_{k_i})| < \varepsilon$, for some $i \in \{1, 2, 3\}$.

Proof. Suppose that this is not the case. Then by using Ramsey theorem [22], we may assume that there exists $\varepsilon > 0$ such that for any $k < \ell < m \in \mathbb{N}$, we have that there exists $f_{k,\ell,m}$ a functional of type II, that separates x_k, x_ℓ, x_m and $|f_{k,\ell,m}(x_k)| > \varepsilon, |f_{k,\ell,m}(x_\ell)| > \varepsilon$ and $|f_{k,\ell,m}(x_m)| > \varepsilon$.

For $1 < k < m$, if $f_{1,k,m} = \frac{1}{2} \sum_{j=1}^{d_{k,m}} f_j^{k,m}$, set

$$\begin{aligned} i_{k,m} &= \min\{j : \text{ran } f_j^{k,m} \cap \text{ran } x_1 \neq \emptyset\} \\ j_{k,m} &= \min\{j : \text{ran } f_j^{k,m} \cap \text{ran } x_k \neq \emptyset\} \\ j'_{k,m} &= \min\{j : \text{ran } f_j^{k,m} \cap \text{ran } x_m \neq \emptyset\} \end{aligned}$$

Notice, that for $1 < k < m$, since $|f_{1,k,m}(x_1)| > \varepsilon$, it follows that

$$\frac{1}{2^{w(f_{i_{k,m}}^{k,m})}} > \frac{\varepsilon}{\|x_1\| \max \text{supp } x_1}$$

By applying Ramsey theorem once more, we may assume that there exists $n_1 \in \mathbb{N}$, such that for any $1 < k < m$, we have that $w(f_{i_{k,m}}^{k,m}) = n_1$

Arguing in the same way and diagonalizing, we may assume that for any $k > 1$, there exists $n_k \in \mathbb{N}$ such that for any $m > k$, we have that $w(f_{j_{k,m}}^{k,m}) = n_k$. Set

$$\begin{aligned} A_1 &= \{\{k, \ell\} \in [\mathbb{N} \setminus \{1\}]^2 : n_k \neq n_\ell \text{ and } n_k \text{ is compatible to } n_\ell\} \\ A_2 &= \{\{k, \ell\} \in [\mathbb{N} \setminus \{1\}]^2 : n_k \neq n_\ell \text{ and } n_k \text{ is not compatible to } n_\ell\} \\ A_3 &= \{\{k, \ell\} \in [\mathbb{N} \setminus \{1\}]^2 : n_k = n_\ell\} \end{aligned}$$

Once more, Ramsey theorem yields that there exists M an infinite subset of the naturals, such that $[M]^2 \subset A_1, [M]^2 \subset A_2$, or $[M]^2 \subset A_3$.

Assume that $[M]^2 \subset A_1$ and for convenience assume that $M = \mathbb{N} \setminus \{1\}$. Choose $k_0 > 1$ such that $k_0 > \max \text{supp } x_1$. Since n_1 is compatible to n_2 and in general n_{k-1} is compatible to n_k , for $k > 1$, it follows that there exists a functional $f = \frac{1}{2} \sum_{j=1}^d f_j$ of type II in W , such that $\text{ran } f \cap \text{ran } x_1 \neq \emptyset$ and for $k = 1, \dots, k_0$ there exists j_k , with $w(f_{j_k}) = n_k$, for $k = 1, \dots, k_0$.

Since $\min \text{supp } f_1 \leq \max \text{supp } x_1$ it follows that $\{f_j\}_{j=1}^d$ can not be \mathcal{S}_1 -admissible, a contradiction.

Assume next that $[M]^2 \subset A_2$. Lemma 4.3 yields that $\beta(\{x_k\}_k) > 0$ and since this cannot be, we conclude that $[M]^2 \subset A_3$, therefore there exists $n_0 \in \mathbb{N}$, such that $n_k = n_0$, for any $k \in M$.

Assume once more that $M = \mathbb{N} \setminus \{1\}$ and set

$$B = \{\{k, \ell, m\} \in [\mathbb{N} \setminus \{1\}]^3 : w(f_{j'_{k,m}}^{1,k,m}) = w(f_{j'_{\ell,m}}^{1,\ell,m})\}$$

If there exists M an infinite subset of the naturals, such that $[M]^3 \subset B^c$, Lemma 4.4 yields that $\beta(\{x_k\}_k) > 0$, therefore by one last Ramsey argument, there exists M an infinite subset of the naturals, such that $[M]^3 \subset B$.

By the above, we conclude that for $m \geq 4$, $\text{ran } x_k \subset \text{ran } f_{j_{2,m}}^{2,m}$ and $|f_{j_{2,m}}^{2,m}(x_k)| > 2\varepsilon$, for $k = 2, \dots, m-2$.

Set $f_m = f_{j_{2,m}}^{2,m}$ and let f be a w^* limit of some subsequence of $\{f_m\}_{m \in \mathbb{N}}$. Then $|f(x_k)| \geq 2\varepsilon$, for any $k \geq 2$. Corollary 2.10 yields a contradiction and this completes the proof. \square

Remark 4.6. The proof of Proposition 4.5 is the only place where the condition $\beta(\{x_k\}_k) = 0$ is needed. This makes necessary to introduce the β -averages and their use in the definition of the norm.

Finite sequences equivalent to ℓ_∞^n basis.

Proposition 4.7. Let $x_1 < \dots < x_n$ be a seminormalized block sequence in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| \leq 1$ for $k = 1, \dots, n$ and there exist $n+3 \leq j_1 < \dots < j_n$ strictly increasing naturals such that the following are satisfied.

- (i) For any $k_0 \in \{1, \dots, n\}$, for any $k \geq k_0, k \in \{1, \dots, n\}$, for any $\{\alpha_q\}_{q=1}^d$ very fast growing and \mathcal{S}_j -admissible sequence of α -averages, with $j < j_{k_0}$ and $s(\alpha_1) > \min \text{supp } x_{k_0}$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < \frac{1}{n \cdot 2^n}$.
- (ii) For any $k_0 \in \{1, \dots, n\}$, for any $k \geq k_0, k \in \{1, \dots, n\}$, for any $\{\beta_q\}_{q=1}^d$ very fast growing and \mathcal{S}_j -admissible sequence of β -averages, with $j < j_{k_0}$ and $s(\beta_1) > \min \text{supp } x_{k_0}$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < \frac{1}{n \cdot 2^n}$.
- (iii) For $k = 1, \dots, n-1$, the following holds: $\frac{1}{2^{j_{k+1}}} \max \text{supp } x_k < \frac{1}{2^n}$.
- (iv) For any $1 \leq k_1 < k_2 < k_3 \leq n$, for any functional $f \in W$ of type II that separates $x_{k_1}, x_{k_2}, x_{k_3}$, we have that $|f(x_{k_i})| < \frac{1}{n \cdot 2^n}$, for some $i \in \{1, 2, 3\}$.

Then $\{x_k\}_{k=1}^n$ is equivalent to ℓ_∞^n basis, with an upper constance $3 + \frac{3}{2^n}$. Moreover, for any functional $f \in W$ of type I_α with weight $w(f) = j < j_1$, we have that $|f(\sum_{k=1}^n x_k)| < \frac{3 + \frac{4}{2^n}}{2^j}$.

Proof. By using Remark 1.3, we will inductively prove, that for any $\{c_k\}_{k=1}^n \subset [-1, 1]$ the following hold.

- (i) For any $f \in W$, we have that $|f(\sum_{k=1}^n c_k x_k)| < (3 + \frac{3}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$.
- (ii) If f is of type I_α and $w(f) \geq 2$, then $|f(\sum_{k=1}^n c_k x_k)| < (1 + \frac{2}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$.
- (iii) If f is of type I_α and $w(f) = j < j_1$, then $|f(\sum_{k=1}^n c_k x_k)| < \frac{3 + \frac{4}{2^n}}{2^j} \max\{|c_k| : k = 1, \dots, n\}$.

For any functional $f \in W_0$ the inductive assumption holds. Assume that it holds for any $f \in W_m$ and let $f \in W_{m+1}$. If f is a convex combination, then there is nothing to prove.

Assume that f is of type I_α , $f = \frac{1}{2^j} \sum_{q=1}^d \alpha_q$, where $\{\alpha_q\}_{q=1}^d$ is a very fast growing and \mathcal{S}_j -admissible sequence of α -averages in W_m .

Set $k_1 = \min\{k : \text{ran } f \cap \text{ran } x_k \neq \emptyset\}$ and $q_1 = \min\{q : \text{ran } \alpha_q \cap \text{ran } x_{k_1} \neq \emptyset\}$.

We distinguish 3 cases.

Case 1: $j < j_1$.

For $q > q_1$, we have that $s(\alpha_q) > \min \text{supp } x_{k_1}$, therefore we conclude that

$$(24) \quad \sum_{q > q_1} |\alpha_q(\sum_{k=1}^n c_k x_k)| < \frac{1}{2^n} \max\{|c_k| : k = 1, \dots, n\}$$

while the inductive assumption yields that

$$(25) \quad |\alpha_{q_1}(\sum_{k=1}^n c_k x_k)| < (3 + \frac{3}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$$

Then (24) and (25) allow us to conclude that

$$(26) \quad |f(\sum_{k=1}^n c_k x_k)| < \frac{3 + \frac{4}{2^n}}{2^j} \max\{|c_k| : k = 1, \dots, n\}$$

Hence, (iii) from the inductive assumption is satisfied.

Case 2: There exists $k_0 < n$, such that $j_{k_0} \leq j < j_{k_0+1}$.

Arguing as previously we get that

$$(27) \quad |f(\sum_{k > k_0} c_k x_k)| < \frac{3 + \frac{4}{2^n}}{2^{j_{k_0}}} \max\{|c_k| : k = 1, \dots, n\}$$

and

$$(28) \quad |f(\sum_{k < k_0} c_k x_k)| < \frac{1}{2^n} \max\{|c_k| : k = 1, \dots, n\}$$

Using (27), (28), the fact that $|f(x_{k_0})| \leq 1$ and $j_{k_0} \geq n + 3$, we conclude that

$$(29) \quad |f(\sum_{k=1}^n c_k x_k)| < (1 + \frac{2}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$$

Case 3: $j \geq j_n$

By using the same arguments, we conclude that

$$(30) \quad |f(\sum_{k=1}^n c_k x_k)| < (1 + \frac{1}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$$

Then (26), (29) and (30) yield that (ii) from the inductive assumption is satisfied.

If f is of type I_β , then the proof is exactly the same, therefore assume that f is of type II, $f = \frac{1}{2} \sum_{j=1}^d f_j$, where $\{f_j\}_{j=1}^d$ is an \mathcal{S}_1 -admissible sequence of functionals of type I_α in W_m . Set

$$E = \{k : |f(x_k)| \geq \frac{1}{n \cdot 2^n}\}$$

$$E_1 = \{k \in E : \text{there exist at least two } j \text{ such that } \text{ran } f_j \cap \text{ran } x_k \neq \emptyset\}$$

Then $\#E_1 \leq 2$. Indeed, if $k_1 < k_2 < k_3 \in E_1$, then f separates x_{k_1}, x_{k_2} and x_{k_3} which contradicts our initial assumptions.

If moreover we set $J = \{j : \text{there exists } k \in E \setminus E_1 \text{ such that } \text{ran } f_j \cap \text{ran } x_k \neq \emptyset\}$, then for the same reasons we get that $\#J \leq 2$.

Since for any j , we have that $w(f_j) \in L$, we get that $w(f_j) > 2$, therefore:

$$(31) \quad |f(\sum_{k \in E \setminus E_1}^n c_k x_k)| < (1 + \frac{2}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$$

$$(32) \quad |f(\sum_{k \in E_1}^n c_k x_k)| \leq 2 \max\{|c_k| : k = 1, \dots, n\}$$

$$(33) \quad |f(\sum_{k \notin E}^n c_k x_k)| \leq n \cdot \frac{1}{n \cdot 2^n} \max\{|c_k| : k = 1, \dots, n\}$$

Finally, (31) to (33) yield the following.

$$|f(\sum_{k=1}^n c_k x_k)| < (3 + \frac{3}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$$

This means that (i) from the inductive assumption is satisfied and this completes the proof. \square

The spreading models of $\mathfrak{X}_{\text{ISP}}$. In this subsection we show that every seminormalized block sequence has a subsequence which generates either ℓ_1 or c_0 as a spreading model.

Proposition 4.8. Let $\{x_k\}_{k \in \mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}_{\text{ISP}}$, such that $\|x_k\| \leq 1$ for all $k \in \mathbb{N}$ and $\alpha(\{x_k\}_k) = 0$ as well as $\beta(\{x_k\}_k) = 0$. Then it has a subsequence, again denoted by $\{x_k\}_{k \in \mathbb{N}}$ satisfying the following.

- (i) $\{x_k\}_{k \in \mathbb{N}}$ generates a c_0 spreading model. More precisely, for any $n \leq k_1 < \dots < k_n$, we have that $\|\sum_{i=1}^n x_{k_i}\| \leq 4$.

- (i) There exists a strictly increasing sequence of naturals $\{j_n\}_{n \in \mathbb{N}}$, such that for any $n \leq k_1 < \dots < k_n$, for any functional f of type I_α with $w(f) = j < j_n$, we have that

$$|f(\sum_{i=1}^n x_{k_i})| < \frac{4}{2^j}$$

Proof. By repeatedly applying Proposition 4.5 and diagonalizing, we may assume that for any $n \leq k_1 < k_2 < k_3$, for any functional f of type II that separates x_{k_1}, x_{k_2} and x_{k_3} , we have that $|f(x_{k_i})| < \frac{1}{n \cdot 2^n}$, for some $i \in \{1, 2, 3\}$.

Use Propositions 3.3 and 3.4 to inductively choose a subsequence of $\{x_k\}_{k \in \mathbb{N}}$, again denoted by $\{x_k\}_{k \in \mathbb{N}}$ and $\{j_k\}_{k \in \mathbb{N}}$ a strictly increasing sequence of naturals with $j_k \geq k + 3$ for all $k \in \mathbb{N}$, such that the following are satisfied.

- (i) For any $k_0 \in \mathbb{N}$, for any $k \geq k_0$, for any $\{\alpha_q\}_{q=1}^d$ very fast growing and \mathcal{S}_j -admissible sequence of α -averages, with $j < j_{k_0}$ and $s(\alpha_1) > \min \text{supp } x_{k_0}$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < \frac{1}{k_0 \cdot 2^{k_0}}$.
- (ii) For any $k_0 \in \mathbb{N}$, for any $k \geq k_0$, for any $\{\beta_q\}_{q=1}^d$ very fast growing and \mathcal{S}_j -admissible sequence of β -averages, with $j < j_{k_0}$ and $s(\beta_1) > \min \text{supp } x_{k_0}$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < \frac{1}{k_0 \cdot 2^{k_0}}$.
- (iii) For $k \in \mathbb{N}$, the following holds: $\frac{1}{2^{j_{k+1}}} \max \text{supp } x_k < \frac{1}{2^k}$.

It is easy to check that for $n \leq k_1 < \dots < k_n$, the assumptions of Proposition 4.7 are satisfied.

□

Propositions 3.5 and 4.8 yield the following.

Corollary 4.9. Let $\{x_k\}_{k \in \mathbb{N}}$ be a normalized weakly null sequence in $\mathfrak{X}_{\text{ISP}}$. Then it has a subsequence that generates a spreading model which is either equivalent to c_0 , or to ℓ_1 .

Proposition 4.10. Let $\{x_k\}_{k \in \mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}_{\text{ISP}}$, that generates a c_0 spreading model. Then there exists $\{F_k\}_{k \in \mathbb{N}}$ an increasing sequence of subsets of the naturals such that $\#F_k \leq \min F_k$ for all $k \in \mathbb{N}$ and $\lim_k \#F_k = \infty$ such that by setting $y_k = \sum_{i \in F_k} x_i$, there exists a subsequence of $\{y_k\}_{k \in \mathbb{N}}$, which generates an ℓ_1^n spreading model, for all $n \in \mathbb{N}$.

In particular, for any $k_0, n \in \mathbb{N}$ and $\varepsilon > 0$, there exists F a finite subset of \mathbb{N} with $\min F \geq k_0$ and $\{c_k\}_{k \in F}$, such that

- (i) $x = \sum_{k \in F} c_k y_k$ is a (n, ε) s.c.c.
- (ii) $\frac{1}{2^n} \leq \|x\| \leq \frac{28}{2^n}$
- (iii) $\{y_k\}_{k \in F}$ is $(4, \{n_k\}_{k \in F})$ α -RIS and $2^{2n} < n_{\min F}$.
- (iv) For any $\eta > 0$ there exists a functional f of type I_α of weight $w(f) = n$ such that $f(x) > \frac{1-\eta}{2^n}$ and $\max \text{supp } f > \max \text{supp } x$.

Proof. Since $\{x_k\}_{k \in \mathbb{N}}$ generates a c_0 spreading model, Proposition 3.5 yields that $\alpha(\{x_k\}_k) = 0$ as well as $\beta(\{x_k\}_k) = 0$, therefore passing, if necessary, to a subsequence $\{x_k\}_{k \in \mathbb{N}}$, satisfies the conclusion of Proposition 4.8.

Choose $\{F_k\}_{k \in \mathbb{N}}$ an increasing sequence of subsets of the naturals, such that the following are satisfied.

- (i) $\#F_k \leq \min F_k$ for all $k \in \mathbb{N}$.
- (ii) $\#F_{k+1} > \max\{\#F_k, (\max \text{supp } x_{\max F_k})^2\}$, for all $k \in \mathbb{N}$.

By Proposition 4.5 and Remark 3.11, we have that $1 \leq \|y_k\| \leq 4$, for all $k \in \mathbb{N}$ and passing, if necessary, to a subsequence, $\{y_k\}_{k \in \mathbb{N}}$ is $(4, \{n_k\}_{k \in \mathbb{N}})$ α -RIS.

Moreover it is easy to see, that for any $k \in \mathbb{N}, \eta > 0$, there exists an α -average α of size $s(\alpha) = \#F_k$, such that $\alpha(y_k) > 1 - \eta$ and $\text{ran } \alpha \subset y_k$.

This yields that $\alpha(\{y_k\}_k) > 0$, therefore we may apply Proposition 3.5 to conclude that $\{y_k\}_{k \in \mathbb{N}}$ has a subsequence generating an ℓ_1^n spreading model, for all $n \in \mathbb{N}$.

We now prove the second assertion. Let $k_0, n \in \mathbb{N}$ and $\varepsilon > 0$. By taking a larger k'_0 , we may assume that $n_{k_0} > 2^{2n}$. Also, by taking a smaller ε , we may assume that $\varepsilon < (160 \cdot 2^{3n})^{-1}$.

Set $\varepsilon' = \varepsilon(1 - \varepsilon)$ Proposition 1.8 yields that there exists $\{d_1, \dots, d_m\}$ a finite subset of $\{k : k \geq k_0\}$ and $\{c_k\}_{k=1}^m$ such that $x' = \sum_{k=1}^m c_k y_{d_k}$ is a (n, ε') s.c.c. It is straightforward to check that $x = \sum_{k=1}^{m-1} \frac{c_k}{1-c_m} y_{d_k}$ is a (n, ε) s.c.c.

By Corollary 2.9 we have that $\|x\| \leq \frac{28}{2^n}$.

For some $\eta > 0$, $k = 1, \dots, m$, choose an α_k -average α_k of size $s(\alpha_k) = \#F_{d_k}$, such that $\alpha_k(y_{d_k}) > 1 - \eta$ and $\text{ran } \alpha_k \subset y_{d_k}$. Set $f = \frac{1}{2^n} (\sum_{k=1}^m \alpha_k)$, which is a functional of type I_α of weight $w(f) = n$ such that $f(x) > \frac{1-\eta}{2^n}$ and $\max \text{supp } f > \max \text{supp } x$.

□

Corollary 4.11. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$. Then Y admits a spreading model equivalent to c_0 as well as a spreading model equivalent to ℓ_1 .

Proof. Assume first that Y is generated by some normalized block sequence $\{x_k\}_{k \in \mathbb{N}}$. Corollary 3.18 and Proposition 4.8 yield that it has a further normalized block sequence $\{y_k\}_{k \in \mathbb{N}}$, generating a spreading model equivalent to c_0 .

Proposition 4.10 yields that $\{y_k\}_{k \in \mathbb{N}}$ has a further block sequence generating an ℓ_1 spreading model.

Since any subspace contains a sequence arbitrarily close to a block sequence, the result follows.

□

We remind that, as Propositions 3.5 and 4.8 state, if a sequence generates an ℓ_1 spreading model, then passing, if necessary, to a subsequence, it generates an ℓ_1^k spreading model for any $k \in \mathbb{N}$. However, as the next proposition states, the space $\mathfrak{X}_{\text{ISP}}$ does not admit higher order c_0 spreading models.

Proposition 4.12. The space $\mathfrak{X}_{\text{ISP}}$ does not admit c_0^2 spreading models.

Proof. Towards a contradiction, assume that there is a sequence $\{x_k\}_{k \in \mathbb{N}}$ in $\mathfrak{X}_{\text{ISP}}$, generating a c_0^2 spreading model. Then it must be weakly null and we may assume that it is a normalized block sequence. By Proposition 4.10, it follows that there exist $\{F_k\}_{k \in \mathbb{N}}$ increasing, Schreier admissible subsets of the naturals and $c > 0$ such that $\|\sum_{j=1}^n \sum_{i \in F_{k_j}} x_i\| \geq n \cdot c$ for any $n \leq k_1 < \dots < k_n$. Since for any such $F_{k_1} < \dots < F_{k_n}$ we have that $\cup_{j=1}^n F_{k_j} \in \mathcal{S}_2$, it follows that $\{x_k\}_{k \in \mathbb{N}}$ does not generate a c_0^2 spreading model. \square

Corollary 4.13. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$. Then Y^* admits a spreading model equivalent to ℓ_1 as well as a spreading model equivalent to c_0^n , for any $n \in \mathbb{N}$.

Proof. Since Y contains a sequence $\{x_k\}_{k \in \mathbb{N}}$ generating a spreading model equivalent to c_0 , which we may assume is Schauder basic, then for any normalized $\{x_k^*\}_{k \in \mathbb{N}} \subset Y^*$, such that $x_k^*(x_m) = \delta_{n,m}$ for $n, m \in \mathbb{N}$, we have that passing, if necessary, to a subsequence, $\{x_k^*\}_{k \in \mathbb{N}}$ generates a spreading model equivalent to ℓ_1 .

To see that Y^* admits a spreading model equivalent to c_0^n for any $n \in \mathbb{N}$, take the previously used sequence $\{x_k\}_{k \in \mathbb{N}}$. Working just like in the proof of Proposition 4.10 find $\{F_k\}_{k \in \mathbb{N}}$ successive subsets of the natural such that $\min F_k \geq \#F_k$, for all $k \in \mathbb{N}$, if $y_k = \sum_{i \in F_k} x_i$ for all $k \in \mathbb{N}$, then $\{y_k\}_{k \in \mathbb{N}}$ is seminormalized and there exists a very fast growing sequence of α -averages $\{\alpha_k\}_{k \in \mathbb{N}} \subset W$ such that $\liminf_k \alpha_k(\sum_{i \in F_k} x_i) \geq 1$.

Then, if $c = \limsup_k \|y_k\|$, we evidently have that $\liminf_k \|\alpha_k\| \geq 1/c$ and since for any $n \in \mathbb{N}$, $F \in \mathcal{S}_n$, we have that $\frac{1}{2^n} \sum_{q \in F} \alpha_q$ is a functional of type I_α in W , it follows that $\|\sum_{q \in F} \alpha_q\| \leq 2^n$. This means that, $\{\alpha_k\}_{k \in \mathbb{N}}$ generates a spreading model equivalent to c_0^n , with an upper constant 2^n .

Let $I^* : \mathfrak{X}_{\text{ISP}}^* \rightarrow Y^*$ be the dual operator of $I : Y \rightarrow \mathfrak{X}_{\text{ISP}}$. Then $\{I^*(\alpha_k)\}_{k \in \mathbb{N}}$ generates a spreading model equivalent to c_0^n , for any $n \in \mathbb{N}$. Since $\|I^*\| = 1$, all that needs to be shown is that $\liminf_k \|I^*(\alpha_k)\| > 0$. Indeed,

$$\liminf_k \|I^*(\alpha_k)\| \geq \liminf_k (I^* \alpha_k) \left(\frac{\sum_{i \in F_k} x_i}{c} \right) = \liminf_k \alpha_k \left(\frac{\sum_{i \in F_k} x_i}{c} \right) \geq 1/c$$

\square

5. PROPERTIES OF $\mathfrak{X}_{\text{ISP}}$ AND $\mathcal{L}(\mathfrak{X}_{\text{ISP}})$

In this final section it is proved that $\mathfrak{X}_{\text{ISP}}$ is hereditarily indecomposable and the properties of the operators acting on infinite dimensional closed subspaces of $\mathfrak{X}_{\text{ISP}}$ are presented.

Dependent sequences and the HI property of $\mathfrak{X}_{\text{ISP}}$. In the first part of this subsection we introduce the dependent sequences, which are the main tool for proving the HI property of $\mathfrak{X}_{\text{ISP}}$ and studying the structure of the operators.

Definition 5.1. A sequence of pairs $\{(x_k, f_k)\}_{k=1}^n$, where $x_1 < \dots < x_n \in \mathfrak{X}_{\text{ISP}}$ and $f_1 < \dots < f_n \in W$, is said to be a 1-dependent sequence (respectively a 0-dependent sequence) if the following are satisfied.

- (i) $\{f_k\}_{k=1}^n$ is an \mathcal{S}_1 -admissible special sequence of type I_α functionals, $f_k(x_k) = 1$ when $\{(x_k, f_k)\}_{k=1}^n$ is 1-dependent (respectively $f_k(x_k) = 0$ when $\{(x_k, f_k)\}_{k=1}^n$ is 0-dependent), $\max \text{supp } x_k < \max \text{supp } f_k$ for $k = 1, \dots, n$ and $\text{ran } f_k \cap \text{ran } x_m = \emptyset$ for $k \neq m$.
- (ii) $x_k = \theta_k 2^{m_k} \sum_{i \in F_k} c_i^k y_i^k$, where $\sum_{i \in F_k} c_i^k y_i^k$ is a (m_k, ε_k) s.c.c. with $m_k = w(f_k)$ and $\frac{1}{28} \leq \theta_k < \frac{29}{28}$, for $k = 1, \dots, n$.
- (iii) $\{y_i^k\}_{i \in F_k}$ is $(4, \{n_i\}_{i \in F_k})$ α -RIS with $2^{2m_k} < n_{\min F_k}$.
- (iv) $\min \text{supp } x_k > 240n2^n 2^{2m_k}$ and $\varepsilon_k < (320n2^n 2^{3m_k})^{-1}$.
- (v) $2^{m_1} > 60n2^n$ and if $p_0 = \min \text{supp } x_1$, then $\frac{1}{p_0} + \sum_{p > p_0} \frac{1}{p^2} < \frac{1}{280n2^n}$.

Proposition 5.2. Let $\{(x_k, f_k)\}_{k=1}^{2n}$ be a 1-dependent sequence in $\mathfrak{X}_{\text{ISP}}$ and set $y_k = x_{2k-1} - x_{2k}$, for $k = 1, \dots, n$. Then we have that:

- (i) $\frac{1}{n} \left\| \sum_{k=1}^{2n} x_k \right\| \geq 1$
- (ii) $\frac{1}{n} \left\| \sum_{k=1}^n y_k \right\| \leq \frac{232}{n}$

Proof. Since $\frac{1}{2} \sum_{k=1}^{2n} f_k$ is a type II functional in W , it immediately follows that $\frac{1}{n} \left\| \sum_{k=1}^{2n} x_k \right\| \geq \frac{1}{2n} \sum_{k=1}^{2n} f_k(x_k) = 1$.

By Corollary 2.9 it follows that $1 = f_k(x_k) \leq \|x_k\| \leq 7 \cdot 4\theta_k \leq 29$ and this yields that $1 \leq \|y_k\| \leq 58$, for $k = 1, \dots, n$. Set $j_k = m_{2k-1} - 2$. We will show that the assumptions of Proposition 4.7 are satisfied. From this, it will follow that $\frac{1}{n} \left\| \sum_{k=1}^n y_k \right\| \leq 58 \frac{4}{n}$, which is the desired result.

The first and second assumptions, follow from Lemmas 3.7 and 3.16 respectively and the definition of the 1-dependent sequence.

The third assumption follows from the fact that, by the definition of the 1-dependent sequence, $\max \text{supp } f_k > \max \text{supp } x_k$, for $k = 1, \dots, 2n$ and the definition of the coding function σ .

It remains to be proven that the fourth assumption is also satisfied. Let $1 \leq k_1 < k_2 < k_3 \leq n$ and $g = \frac{1}{2} \sum_{j=1}^d g_j$ be a functional of type II that separates y_{k_1}, y_{k_2} and y_{k_3} .

Set $j_0 = \min\{j : \text{ran } g_j \cap \text{ran } y_{k_3} \neq \emptyset\}$ and assume first that $w(f_{j_0}) = m_{2k_3-1}$. Since $\text{supp } g \cap \text{supp } y_{k_1} \neq \emptyset$, it follows that $g_{j_0-1} = f_{2k_3-2}$ and there

exists I an interval of the naturals, $\text{ran } y_{k_2} \subset I$, such that $g = I(\frac{1}{2} \sum_{k=1}^{j_0-1} f_k)$. This yields that $g(y_{k_2}) = 0$.

Otherwise, if $w(f_{j_0}) \neq m_{2k_3-1}$, set $g' = g|_{\text{ran } y_{k_3}}$ and Corollary 3.14 yields the following.

$$|g'(y_{k_3})| < 2 \cdot 4 \frac{29}{28} \left(\frac{1}{2^{m_{2k_3-1}}} + \frac{1}{2^{2m_{2k_3-1}}} + \sum_{\substack{j \in \widehat{w}(g'): \\ w(g_j) < n}} \frac{4}{2^{w(g_j)}} + 2^{m_{2k_3-1}} \varepsilon_{m_{2k_3-1}} \right)$$

Since g separates y_{k_1}, y_{k_2} and y_{k_3} , we have that $\min \widehat{w}(g') \geq p_0 = \min \text{supp } x_1$, therefore

$$\sum_{\substack{j \in \widehat{w}(g'): \\ w(g_j) < n}} \frac{1}{2^{w(g_j)}} < \sum_{p \geq p_0} \frac{1}{2^p} < \sum_{p \geq p_0} \frac{1}{p^2} < \frac{1}{280 \cdot 2n2^{2n}}$$

Moreover, we have that

$$\frac{1}{2^{m_{2k_3-1}}} + \frac{1}{2^{2m_{2k_3-1}}} + 2^{m_{2k_3-1}} \varepsilon_{m_{2k_3-1}} < \frac{3}{2^{m_1}} < \frac{1}{20 \cdot 2n2^{2n}}$$

We conclude that $|g(y_{k_3})| < \frac{1}{2n2^{2n}} < \frac{1}{n2^n}$, which means that the fourth assumption is satisfied. \square

The next proposition is proved by using similar arguments.

Proposition 5.3. Let $\{(x_k, f_k)\}_{k=1}^n$ be a 0-dependent sequence in $\mathfrak{X}_{\text{ISP}}$. Then we have that:

$$\frac{1}{n} \left\| \sum_{k=1}^n x_k \right\| \leq \frac{112}{n}$$

We pass to the main structural property of $\mathfrak{X}_{\text{ISP}}$.

Theorem 5.4. The space $\mathfrak{X}_{\text{ISP}}$ is hereditarily indecomposable.

Proof. It is enough to show that for X, Y block subspaces of $\mathfrak{X}_{\text{ISP}}$, for any $\varepsilon > 0$, there exist $x \in X$ and $y \in Y$, such that $\|x + y\| \geq 1$ and $\|x - y\| < \varepsilon$. Let $n \in \mathbb{N}$, such that $\frac{232}{n} < \varepsilon$.

By Corollary 3.18 and Proposition 4.8, there exist $\{x_k\}_{k \in \mathbb{N}}$ a normalized block sequence in X and $\{y_k\}_{k \in \mathbb{N}}$ a normalized block sequence in Y , both generating c_0 spreading models.

Choose $m_1 \in L_1$ (see the definition of the coding function) such that $2^{m_1} > 60 \cdot 2n2^{2n}$, $p_0 \in \mathbb{N}$ such that $\frac{1}{p_0} + \sum_{p > p_0} \frac{1}{p^2} < \frac{1}{280 \cdot 2n2^{2n}2^{2m_1}}$ and $0 < \varepsilon_1 < (320 \cdot 2n2^{2n}2^{3m_1})^{-1}$.

By Proposition 4.10 there exists $x''_1 = \sum_{i \in F_1} c_i^1 x_i$ a (m_1, ε_1) s.c.c. and f_1 a functional of type I_α such that

- (i) $\frac{1}{2^{m_1}} \leq \|x''_1\| \leq \frac{28}{2^{m_1}}$
- (ii) $\{x_k\}_{k \in F_1}$ is $(4, \{n_k\}_{k \in F_1})$ α -RIS and $2^{2m_1} < n_{\min F}$.
- (iii) f_1 is of weight $w(f_1) = m_1$ and $f_1(x''_1) > \frac{28}{29 \cdot 2^{m_1}}$ and $\max \text{supp } f > \max \text{supp } x''_1$.

Set $\theta_1 = (2^{m_1} f_1(x_1''))^{-1}$ and $x_1' = \theta_1 2^{m_1} \sum_{i \in F_1} c_i^1 x_i$.

Continue in the same manner and construct x_1', \dots, x_{2n}' and f_1, \dots, f_{2n} such that $\{(x_k', f_k)\}_{k=1}^{2n}$ is a 1-dependent sequence and $x_{2k-1}' \in X, x_{2k}' \in Y$, for $k = 1, \dots, n$.

Set $x = \frac{1}{n} \sum_{k=1}^n x_{2k-1}'$ and $y = \frac{1}{n} \sum_{k=1}^n x_{2k}'$. By applying Proposition 5.2, the result follows. \square

The structure of $\mathcal{L}(Y, \mathfrak{X}_{\text{ISP}})$. For Y a closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \rightarrow \mathfrak{X}_{\text{ISP}}$ we show that $T = \lambda I_{Y, \mathfrak{X}_{\text{ISP}}} + S$ with $S : Y \rightarrow \mathfrak{X}_{\text{ISP}}$ strictly singular.

Proposition 5.5. Let Y be a subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \rightarrow \mathfrak{X}_{\text{ISP}}$ be a linear operator, such that there exists $\{x_k\}_{k \in \mathbb{N}}$ a sequence in Y generating a c_0 spreading model and $\limsup \text{dist}(Tx_k, \mathbb{R}x_k) > 0$. Then T is unbounded.

Proof. Passing, if necessary, to a subsequence, there exists $1 > \delta > 0$, such that $\text{dist}(Tx_k, \mathbb{R}x_k) > \delta$, for any $k \in \mathbb{N}$.

Since $\{x_k\}_{k \in \mathbb{N}}$ generates a c_0 spreading model, it is weakly null. Set $y_k = Tx_k$ and assume that T is bounded. It follows that passing, if necessary, to a subsequence of $\{x_k\}_{k \in \mathbb{N}}$, then $\{y_k\}_{k \in \mathbb{N}}$ also generates a c_0 spreading model.

We may assume that $\{x_k\}_{k \in \mathbb{N}}$, as well as $\{y_k\}_{k \in \mathbb{N}}$ are block sequences with rational coefficients. And $\lim_k \|x_k\| = 1$, as well as $\lim_k \|y_k\| = 1$.

If this is not the case pass, if necessary, to a further subsequence of $\{x_k\}_{k \in \mathbb{N}}$, such that both $\{x_k\}_{k \in \mathbb{N}}$ and $\{y_k\}_{k \in \mathbb{N}}$ are equivalent to some block sequences with rational coefficients $\{x_k'\}_{k \in \mathbb{N}}, \{y_k'\}_{k \in \mathbb{N}}$ respectively, and moreover $\lim_k \|x_k'\| = 1$, as well as $\lim_k \|y_k'\| = 1$. Set $Y' = [\{x_k'\}_{k \in \mathbb{N}}]$ and $T' : Y' \rightarrow \mathfrak{X}_{\text{ISP}}$, such that $T'x_k' = y_k'$. It is easy to check that T' is also bounded and $\text{dist}(T'x_k', \mathbb{R}x_k') > \delta'$, for some $\delta' > 0$.

Set $I_k = \text{ran}(\text{ran } x_k \cup \text{ran } y_k)$ and passing, if necessary, to a subsequence, we have that $\{I_k\}_{k \in \mathbb{N}}$ is an increasing sequence of intervals of the naturals.

We will choose $\{f_k\}_{k \in \mathbb{N}} \subset W$, such that $f_k(y_k) > \frac{\delta}{5}$, $f_k(x_k) = 0$ and $\text{ran } f_k \subset I_k$, for all $k \in \mathbb{N}$.

The Hahn-Banach Theorem, yields that for all $k \in \mathbb{N}$, there exists $f_k' \in B_{\mathfrak{X}_{\text{ISP}}^*}$, such that $f_k'(y_k) > \delta$, $f_k'(x_k) = 0$ and $\text{ran } f_k' \subset I_k$, for all $k \in \mathbb{N}$.

By the fact that $\mathfrak{X}_{\text{ISP}}$ is reflexive, it follows that W is norm dense in $B_{\mathfrak{X}_{\text{ISP}}^*}$, therefore there exists $f_k'' \in W$ with $\|f_k' - f_k''\| < \frac{\delta}{4}$ and $\text{ran } f_k'' \subset I_k$, for all $k \in \mathbb{N}$.

It follows that $f_k''(y_k) > \frac{3\delta}{4}$, $|f_k''(x_k)| < \frac{\delta}{4}$ and $f_k''(x_k)$ is rational, for all $k \in \mathbb{N}$.

Furthermore, there exists $g_k \in W$, such that $g_k(x_k) > 1 - \frac{\delta}{4}$, $g_k(x_k)$ is rational and $\text{ran } g_k \subset I_k$, for all $k \in \mathbb{N}$.

Set $f_k = \frac{1}{2}(f_k'' - \frac{f_k''(x_k)}{g_k(x_k)}g_k)$. By doing some easy calculations, it follows that the f_k are the desired functionals.

By copying the proof of Proposition 4.10, for any $k_0 \in \mathbb{N}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists F a finite subset of the naturals with $\min F \geq k_0$ and $\{c_k\}_{k \in F}$ such that

- (i) $\sum_{k \in F} c_k x_k$ is a (n, ε) s.c.c.
- (ii) If $z = 2^n \sum_{k \in F} c_k x_k$, then $\frac{28}{29} \leq \|z\| \leq 28$.
- (iii) $\{x_k\}_{k \in F}$ is $(4, \{n_k\}_{k \in F})$ α -RIS and $2^{2^n} < n_{\min F}$.
- (iv) There exists a functional f of type I_α with weight $w(f) = n$ such that $f(z) = 0$, $\max \text{supp } f > \max \text{supp } z$ and if $w = 2^n \sum_{k \in F} c_k y_k$, then $f(w) > \frac{\delta}{5}$.

Arguing in the same way as in the proof of Theorem 5.4, for some $n \in \mathbb{N}$, we construct a sequence $\{z_k\}_{k=1}^n$ and $\{g_k\}_{k=1}^n$ such that $\{(z_k, g_k)\}_{k=1}^n$ is 0-dependent and if $w_k = Tz_k$, then $g_k(w_k) > \frac{\delta}{5}$ and $\text{ran } g_k \cap \text{ran } w_m = \emptyset$, for $k \neq m$.

Then $f = \frac{1}{2} \sum_{k=1}^n g_k$ is a functional of type II and $\frac{1}{n} \|\sum_{k=1}^n w_k\| \geq \frac{1}{2n} \sum_{k=1}^n g_k(w_k) > \frac{\delta}{10}$.

Moreover, Proposition 5.3 yields that $\frac{1}{n} \|\sum_{k=1}^n z_k\| \leq \frac{112}{n}$. It follows that $\|T\| > \frac{n\delta}{1120}$. Since n was randomly chosen, T cannot be bounded, a contradiction which completes the proof. \square

In [14], it is proven that if X is a hereditarily indecomposable complex Banach space, Y is a subspace of X and $T : Y \rightarrow X$ is a bounded linear operator, then there exists $\lambda \in \mathbb{C}$, such that $T - \lambda I_{Y,X} : Y \rightarrow X$ is strictly singular. Here we prove a similar result for $\mathfrak{X}_{\text{ISP}}$.

Theorem 5.6. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \rightarrow \mathfrak{X}_{\text{ISP}}$ be a bounded linear operator. Then there exists $\lambda \in \mathbb{R}$, such that $T - \lambda I_{Y, \mathfrak{X}_{\text{ISP}}} : Y \rightarrow \mathfrak{X}_{\text{ISP}}$ is strictly singular.

Proof. If T is strictly singular, then evidently $\lambda = 0$ is the desired scalar.

Otherwise, choose Z an infinite dimensional closed subspace of Y , such that $T : Z \rightarrow \mathfrak{X}_{\text{ISP}}$ is an into isomorphism. Choose $\{x_k\}_{k \in \mathbb{N}}$ a normalized sequence in Z generating a c_0 spreading model. Proposition 5.5 yields that $\lim_k \text{dist}(Tx_k, \mathbb{R}x_k) = 0$. Choose $\{\lambda_k\}_{k \in \mathbb{N}}$ scalars, such that $\lim_k \|Tx_k - \lambda_k x_k\| = 0$ and λ a limit point of $\{\lambda_k\}_{k \in \mathbb{N}}$.

We will prove that $S = T - \lambda I_{Y, \mathfrak{X}_{\text{ISP}}}$ is strictly singular. Towards a contradiction, suppose that this is not the case. Then there exists $\{y_k\}_{k \in \mathbb{N}}$ a normalized sequence in Y generating a c_0 spreading model and $\delta > 0$, such that $\|Sy_k\| = \|(T - \lambda I_{Y, \mathfrak{X}_{\text{ISP}}})y_k\| > \delta$, for all $k \in \mathbb{N}$.

As previously, we may assume that $\{x_k\}_{k \in \mathbb{N}}$, $\{y_k\}_{k \in \mathbb{N}}$ as well as $\{Sy_k\}_{k \in \mathbb{N}}$ are all normalized block sequences generating c_0 spreading models.

By Proposition 5.5 and passing, if necessary, to a subsequence, there exists $\mu \in \mathbb{R}$, such that $\lim_k \|Sy_k - \mu y_k\| = 0$. Evidently $\mu \neq 0$, otherwise we would

have that $\lim_k \|Sy_k\| = 0$. Pass, if necessary, to a further subsequence of $\{y_k\}_{k \in \mathbb{N}}$, such that $\sum_{k=1}^{\infty} \|Sy_k - \mu y_k\| < \frac{|\mu|}{232}$.

Observe that $\lim_k \|Sx_k\| = 0$ and therefore we may pass, if necessary, to a subsequence of $\{x_k\}_{k \in \mathbb{N}}$, such that $\sum_{k=1}^{\infty} \|Sx_k\| < \frac{|\mu|}{232}$.

Arguing in the same manner as in the proof of Theorem 5.4, for some $n \in \mathbb{N}$ construct $\{z_k\}_{k=1}^{2n}$ and $\{f_k\}_{k=1}^{2n}$ such that z_{2k-1} is a linear combination of $\{y_k\}_{k \in \mathbb{N}}$, z_{2k} is a linear combination of $\{x_k\}_{k \in \mathbb{N}}$ and $\{(z_k, f_k)\}_{k=1}^{2n}$ is a 1-dependent sequence. Set $f = \frac{1}{2} \sum_{k=1}^{2n} f_k$, which is a functional of type II in W .

If $w_k = z_{2k-1} - z_{2k}$, Proposition 5.2 yields that $\frac{1}{n} \|\sum_{k=1}^n w_k\| \leq \frac{232}{n}$.

On the other hand, we have that

$$\begin{aligned} \frac{1}{n} \left\| \sum_{k=1}^n Sw_k \right\| &\geq \frac{1}{n} \left(\left\| \sum_{k=1}^n Sz_{2k-1} \right\| - \left\| \sum_{k=1}^n Sz_{2k} \right\| \right) \\ &\geq \frac{1}{n} \left(\left\| \sum_{k=1}^n \mu z_{2k-1} \right\| - \left\| \sum_{k=1}^n (Sz_{2k-1} - \mu z_{2k-1}) \right\| - \frac{29|\mu|}{232} \right) \\ &\geq \frac{1}{n} \left(\frac{|\mu|}{2} \sum_{k=1}^n f_{2k-1}(z_{2k-1}) - \frac{29|\mu|}{232} - \frac{29|\mu|}{232} \right) \\ &= \frac{|\mu|}{2} - \frac{|\mu|}{4n} \geq \frac{|\mu|}{4} \end{aligned}$$

It follows that $\|S\| \geq \frac{n|\mu|}{928}$, where n was randomly chosen. This means that S is unbounded, a contradiction completing the proof. \square

Strictly Singular Operators. In this subsection we study the action of strictly singular operators on Schauder basic sequences in subspaces of $\mathfrak{X}_{\text{ISP}}$.

Proposition 5.7. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \rightarrow \mathfrak{X}_{\text{ISP}}$ be a linear bounded operator. Then the following assertions are equivalent.

- (i) T is not strictly singular.
- (ii) There exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ in Y generating a c_0 spreading model, such that $\{Tx_k\}_{k \in \mathbb{N}}$ is not norm convergent to 0.

Proof. Assume first that T is not strictly singular and let Z be an infinite dimensional closed subspace of Y , such that $T|_Z$ is an isomorphism. Since any subspace of $\mathfrak{X}_{\text{ISP}}$ contains a sequence generating a c_0 spreading model, then so does Z . Since $T|_Z$ is an isomorphism, the second assertion is true.

Assume now that there exists $\{x_k\}_{k \in \mathbb{N}}$ a sequence in Y generating a c_0 spreading model, such that $\{Tx_k\}_{k \in \mathbb{N}}$ does not norm converge to 0. By Proposition 5.5 and passing, if necessary to a subsequence, there exists $\lambda \neq 0$, such that $\lim_k \|Tx_k - \lambda x_k\| = 0$. Passing, if necessary, to a further

subsequence, we have that $\sum_{k=1}^{\infty} \|Tx_k - \lambda x_k\| < \infty$. But this means that $\{x_k\}_{k \in \mathbb{N}}$ is equivalent to $\{Tx_k\}_{k \in \mathbb{N}}$, therefore T is not strictly singular. \square

Proposition 5.8. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $S, T : Y \rightarrow Y$ be strictly singular operators. Let also $\{x_k\}_{k \in \mathbb{N}}$ be a weakly null sequence in Y generating an ℓ_1 spreading model, such that $\{STx_k\}_{k \in \mathbb{N}}$ does not norm converge to 0. Then passing, if necessary, to a subsequence, $\{STx_k\}_{k \in \mathbb{N}}$ generates a c_0 spreading model.

Proof. Towards a contradiction, assume that this is not the case. Then passing, if necessary to a subsequence, $\{x_k\}_{k \in \mathbb{N}}$, $\{Tx_k\}_{k \in \mathbb{N}}$ and $\{STx_k\}_{k \in \mathbb{N}}$ all generate ℓ_1 spreading models.

Set $y_k = Tx_k$ and $z_k = STx_k$. We may assume that $\{x_k\}, \{y_k\}_{k \in \mathbb{N}}$ as well as $\{z_k\}_{k \in \mathbb{N}}$ are all seminormalized block sequences. Set $I_k = \text{ran}(\text{ran } x_k \cup \text{ran } y_k \cup \text{ran } z_k)$ for all $k \in \mathbb{N}$ and pass, if necessary, to a subsequence of $\{x_k\}_{k \in \mathbb{N}}$, such that $\{I_k\}_{k \in \mathbb{N}}$ is an increasing sequence of intervals of the naturals and set $j_k = \min I_k$.

It is not hard to see that if $\sum_{k \in F} c_k e_{j_k}$ is a (n, ε) basic s.c.c. then $\sum_{k \in F} c_k x_k, \sum_{k \in F} c_k y_k$ as well as $\sum_{k \in F} c_k z_k$, are all $(n, 2\varepsilon)$ s.c.c.

Moreover either $\alpha(\{x_k\}_k) > 0$ or $\beta(\{x_k\}_k) > 0$ and the same is true for $\{y_k\}_{k \in \mathbb{N}}$ and $\{z_k\}_{k \in \mathbb{N}}$.

Arguing in the same way as in the proof of Proposition 3.5, there exists $c > 0$, such that for any $n, k_0 \in \mathbb{N}$ and $\varepsilon > 0$, there exists F a finite subset of the naturals with $\min F \geq k_0$ and $\{c_k\}_{k \in F}$, such that

- (i) $\sum_{k \in F} c_k x_k, \sum_{k \in F} c_k y_k$, and $\sum_{k \in F} c_k z_k$ are (n, ε) s.c.c.
- (ii) $\|\sum_{k \in F} c_k x_k\| > \frac{c}{2^n}, \|\sum_{k \in F} c_k y_k\| > \frac{c}{2^n}$, and $\|\sum_{k \in F} c_k z_k\| > \frac{c}{2^n}$

Choose $\{n_k\}_{k \in \mathbb{N}}$ strictly increasing naturals, $\{F_k\}_{k \in \mathbb{N}}$ increasing subsets of the naturals and $\{c_i\}_{i \in F_k}$, such that if $x'_k = 2^{n_k} \sum_{i \in F_k} c_i x_i, Tx'_k = y'_k = 2^{n_k} \sum_{i \in F_k} c_i y_i$ and $STx'_k = z'_k = 2^{n_k} \sum_{i \in F_k} c_i z_i$, then $\{x'_k\}_{k \in \mathbb{N}}, \{y'_k\}_{k \in \mathbb{N}}$ and $\{z'_k\}_{k \in \mathbb{N}}$ are all seminormalized and satisfy the assumptions of Proposition 3.8. Therefore $\alpha(\{x'_k\}_k) = 0, \alpha(\{y'_k\}_k) = 0$ and $\alpha(\{z'_k\}_k) = 0$.

We will show that $\beta(\{x'_k\}_k) > 0$, as well as $\beta(\{y'_k\}_k) > 0$. If $\beta(\{x'_k\}_k) = 0$, then Proposition 4.8 yields that passing, if necessary, to a subsequence, $\{x'_k\}_{k \in \mathbb{N}}$ generates a c_0 spreading model and since $\{y'_k\}_{k \in \mathbb{N}}$ is seminormalized and $Tx'_k = y'_k$, so does $\{y'_k\}_{k \in \mathbb{N}}$. Proposition 5.7 yields a contradiction. For the same reasons we also conclude that $\beta(\{y'_k\}_k) > 0$.

Using the same arguments, we may construct $\{w_k\}_{k \in \mathbb{N}}$ a block sequence of $\{x'_k\}_{k \in \mathbb{N}}$, such that both $\{w_k\}_{k \in \mathbb{N}}$ and $\{Tw_k\}_{k \in \mathbb{N}}$ are seminormalized and satisfy the assumptions of Corollary 3.17. Therefore we conclude that $\alpha(\{w_k\}_k) = 0, \beta(\{w_k\}_k) = 0, \alpha(\{Tw_k\}_k) = 0$ and $\beta(\{Tw_k\}_k) = 0$. Propositions 4.8 and 5.7 yield a contradiction and this completes the proof. \square

The Invariant Subspace Property.

Theorem 5.9. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $Q, S, T : Y \rightarrow Y$ be strictly singular operators. Then QST is compact.

Proof. Since $\mathfrak{X}_{\text{ISP}}$ is reflexive, it is enough to show that for any weakly null sequence $\{x_k\}_{k \in \mathbb{N}}$, we have that $\{QSTx_k\}_{k \in \mathbb{N}}$ norm converges to zero. Pass, if necessary, to a subsequence again denoted by $\{x_k\}_{k \in \mathbb{N}}$, that generates some spreading model, which is, as we have shown, either equivalent to ℓ_1 , or to c_0 .

Assume first $\{x_k\}_{k \in \mathbb{N}}$ generates a c_0 spreading model. If $\{Tx_k\}_{k \in \mathbb{N}}$ is not norm convergent, then it has a subsequence generating a c_0 spreading model as well. Proposition 5.7 yields a contradiction.

If $\{x_k\}_{k \in \mathbb{N}}$ generates an ℓ_1 spreading model and $\{STx_k\}_{k \in \mathbb{N}}$ is not norm convergent, then Proposition 5.8 yields that passing, if necessary, to a subsequence of $\{x_k\}_{k \in \mathbb{N}}$, $\{STx_k\}_{k \in \mathbb{N}}$ will generate a c_0 spreading model. Arguing as in the previous case, we conclude that $\{QSTx_k\}_{k \in \mathbb{N}}$ is norm convergent and this completes the proof. \square

Corollary 5.10. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $S : Y \rightarrow Y$ be a non zero strictly singular operator. Then S admits a non-trivial closed hyperinvariant subspace.

Proof. Assume first that $S^3 = 0$. Then it is straightforward to check that $\ker S$ is a non-trivial closed hyperinvariant subspace of S .

Otherwise, if $S^3 \neq 0$, then Theorem 5.9 yields that S^3 is compact and non zero. Since S commutes with its cube, by Theorem 2.1 from [27], it is enough to check that for any $\alpha, \beta \in \mathbb{R}$ such that $\beta \neq 0$, we have that $(\alpha I - S)^2 + \beta^2 I \neq 0$. Since S is strictly singular, it is easy to see that this condition is satisfied. \square

Corollary 5.11. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$ and $T : Y \rightarrow Y$ be a non scalar operator. Then T admits a non-trivial closed hyperinvariant subspace.

Proof. Theorem 5.6 yields that there exist $\lambda \in \mathbb{R}$, such that $S = T - \lambda I$ is strictly singular, and since T is not a scalar operator, we evidently have that S is not zero.

By Corollary 5.10, it follows that S admits a non-trivial closed hyperinvariant subspace Z . It is straightforward to check that Z also is a hyperinvariant subspace for T . \square

In the final result, which is related to Proposition 3.1 from [5], we show that the “scalar plus compact” property fails in every subspace of $\mathfrak{X}_{\text{ISP}}$.

Proposition 5.12. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$. Then there exists a strictly singular, non compact operator $S : Y \rightarrow Y$. In fact, if $\mathcal{S}(Y)$ is the space of strictly singular operators on Y , then $\mathcal{S}(Y)$ is non-separable.

Proof. By Corollary 4.11, there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ in Y that generates a spreading model equivalent to c_0 , say with an upper constant c_1 and by Corollary 4.13, there exists a sequence $\{x_k^*\}_{k \in \mathbb{N}}$ in Y^* that also generates a spreading model equivalent to c_0 , say with an upper constant c_2 . Therefore $\{x_k\}_{k \in \mathbb{N}}$ and $\{x_k^*\}_{k \in \mathbb{N}}$ are weakly null and we may assume that they are Schauder basic and that $\dim(Y/[x_k]_k) = \infty$. We may also assume that there exist $\{z_k\}_{k \in \mathbb{N}}$ in Y such that $\{x_k^*\}_{k \in \mathbb{N}}$ is almost biorthogonal to $\{z_k\}_{k \in \mathbb{N}}$.

For $\varepsilon > 0$, set $M_\varepsilon = \frac{4c_1}{\varepsilon}$. Choose a strictly increasing sequence of naturals $\{q_j\}_{j \in \mathbb{N}}$, such that $q_j \geq M_{1/2^{j+1}}$. Set $S : Y \rightarrow Y$, such that $Sx = \sum_{k=1}^{\infty} x_{q_k}^*(x)x_k$. Then:

- (i) S is bounded and non compact.
- (ii) S is strictly singular.

We first prove that it is bounded. Let $x \in Y, \|x\| = 1, x^* \in Y^*, \|x^*\| = 1$. For $j \geq 0$, set $B_j = \{k \in \mathbb{N} : 1/2^{j+1} < |x^*(x_k)| \leq 1/2^j\}$. Since $\{x_k\}_{k \in \mathbb{N}}$ generates c_0 as a spreading model, it follows that $B_j \leq M_{1/2^{j+1}} \leq q_j$. Set $C_j = \{k \in B_j : k \geq j\}, D_j = B_j \setminus C_j$. Evidently $\#D_j \leq j$ and it is easy to see that $\#\{q_k : k \in C_j\} \leq \min\{q_k : k \in C_j\}$, therefore, since $\{x_k^*\}_{k \in \mathbb{N}}$ generates a spreading model equivalent to c_0 , it follows that

$$\left| \sum_{k \in C_j} x^*(x_k)x_{q_k}^*(x) \right| < c_2 \max\{|x^*(x_k)| : k \in C_j\}$$

Therefore $|\sum_{k \in B_j} x_{q_k}^*(x)x^*(x_k)| \leq c_2 \max\{|x^*(x_k)| : k \in C_j\} + j/2^j \leq c_2/2^j + j/2^j$. From this it follows that

$$\|Sx\| \leq \sum_{j=0}^{\infty} \frac{j + c_2}{2^j} \|x\|$$

The fact that S is non compact follows easily if you consider the almost biorthogonals $\{z_k\}_{k \in \mathbb{N}}$ of $\{x_{q_k}^*\}_{k \in \mathbb{N}}$. Then $\{z_k\}_{k \in \mathbb{N}}$ is a seminormalized sequence in Y and $\{Sz_k\}_{k \in \mathbb{N}}$ does not have a norm convergent subsequence.

We now prove that S is strictly singular. Suppose that it is not, then there exists $\lambda \neq 0$ such that $T = S - \lambda I$ is strictly singular. Since λI is a Fredholm operator and T is strictly singular, it follows that $S = T + \lambda I$ is also a Fredholm operator, therefore $\dim(Y/S[Y]) < \infty$. The fact that $S[Y] \subset [x_k]_k$ and $\dim(Y/[x_k]_k) = \infty$ yields a contradiction.

Moreover, for any further subsequence $\{x_k^*\}_{k \in L}$ of $\{x_{q_k}^*\}_{k \in \mathbb{N}}$, if we set $S_L x = \sum_{k \in L} x_k^*(x)x_k$, then S_L satisfies the same conditions. This yields that $\mathcal{S}(Y)$ contains an uncountable ε -separated set and is therefore non-separable. \square

The last proof actually yields that if Y is an infinite dimensional closed subspace of $\mathfrak{X}_{\text{ISP}}$, then the space of strictly singular, non-compact operators of Y is non-separable.

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